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Solution to problem: 11884

Let $f$ be a real-valued function on $[0,1]$ such that $f$ and its first two derivatives are continuous and $f(\frac{1}{2})=0$.

Let's show that $\int\limits_0^1 (f''(x))^2dx \geq 320\left(\int\limits_0^1 f(x)dx\right)^2$.

Let $I=\int\limits_0^{\frac{1}{2}} \frac{1}{2}x^2(f''(x)+f''(1-x))dx$.

We have $I=\left[\frac{1}{2}x^2(f'(x)-f'(1-x))\right]_0^{\frac{1}{2}}-\int\limits_0^{\frac{1}{2}} x(f'(x)-f'(1-x))dx$

where $\begin{cases} u(x)=\frac{1}{2}x^2 \Rightarrow u'(x)=x \\ v'(x)=f''(x)+f''(1-x) \Rightarrow v(x)=f'(x)-f'(1-x) \end{cases}$

So $I=0-\int\limits_0^{\frac{1}{2}} x(f'(x)-f'(1-x))dx$

$=-\left[x(f(x)+f(1-x))\right]_0^{\frac{1}{2}}+\int\limits_0^{\frac{1}{2}}(f(x)+f(1-x))dx$

where $\begin{cases} u(x)=x \Rightarrow u'(x)=1 \\ v'(x)=f'(x)-f'(1-x) \Rightarrow v(x)=f(x)+f(1-x) \end{cases}$

So $I=0+\int\limits_0^{\frac{1}{2}} f(x)dx+\int\limits_0^{\frac{1}{2}} f(1-x)dx$ because $f(\frac{1}{2})=0$

$=\int\limits_0^{\frac{1}{2}} f(x)dx-\int\limits_1^{\frac{1}{2}} f(u)du$ where $\begin{cases} u=1-x \\ du=-dx \end{cases}$

$=\int\limits_0^1 f(x)dx$

Furthermore $I^2=\left(\int\limits_0^{\frac{1}{2}} \frac{1}{2}x^2(f''(x)+f''(1-x))dx\right)^2$

$\leq \int\limits_0^{\frac{1}{2}}(\frac{1}{2}x^2)^2dx \int\limits_0^{\frac{1}{2}}(f''(x)+f''(1-x))^2dx$ (Cauchy−Schwarz)

$=\left[\frac{1}{4}\frac{1}{5}x^5\right]_0^{\frac{1}{2}} \int\limits_0^{\frac{1}{2}}(f''(x)+f''(1-x))^2dx$

And we have $\forall a, b \in \mathbb{R},\ 0 \leq (a-b)^2 \ \Rightarrow\ \ 2ab \leq a^2+b^2 \ \Rightarrow\ (a+b)^2 \leq 2(a^2+b^2)$.

$$\begin{aligned}
\text{So } I^2 &\leq (\tfrac{1}{4}\tfrac{1}{5}\tfrac{1}{2^5})2\int_0^{\frac{1}{2}}(f''(x))^2+(f''(1-x))^2dx \\
&= \tfrac{1}{320}\left(\int_0^{\frac{1}{2}}(f''(x))^2dx+\int_0^{\frac{1}{2}}(f''(1-x))^2dx\right) \\
&= \tfrac{1}{320}\left(\int_0^{\frac{1}{2}}(f''(x))^2dx-\int_1^{\frac{1}{2}}(f''(u))^2du\right) \qquad \text{where}\begin{cases} u=1-x \\ du=-dx \end{cases} \\
&= \tfrac{1}{320}\int_0^1(f''(x))^2dx
\end{aligned}$$

Therefore $\left(\int_0^1 f(x)dx\right)^2 = I^2 \leq \frac{1}{320}\int_0^1(f''(x))^2dx$

Hence $\int_0^1(f''(x))^2dx \geq 320\left(\int_0^1 f(x)dx\right)^2$.