An Integral Inequality

11918 [2016, 613]. *Proposed by Phu Cuong Le Van, College of Education, Hue University, Hue City, Vietnam.* Let f be n times continuously differentiable on [0, 1], with $f(1/2) = 0$ and $f^{(i)}(1/2) = 0$ when i is even and at most n. Prove

$$\left(\int_0^1 f(x)\,dx\right)^2 \le \frac{1}{(2n+1)2^{2n}(n!)^2}\int_0^1 (f^{(n)}(x))^2\,dx.$$

Solution by Patrick J. Fitzsimmons, University of California, San Diego, La Jolla, CA. Let F be an antiderivative of f. Using Taylor's theorem with remainder in integral form, we expand F in powers of $t - 1/2$ to obtain

$$F(t) = F(1/2) + \sum_{k=0}^{n-1} \frac{f^{(k)}(1/2)}{(k+1)!}\left(t - \frac{1}{2}\right)^{k+1} + \int_{1/2}^{t} \frac{f^{(n)}(x)}{n!}(t-x)^n dx$$

for any t in [0, 1]. In particular, with $t = 1$,

$$\int_{1/2}^1 f(x)\,dx = \sum_{k=0}^{n-1} \frac{f^{(k)}(1/2)}{(k+1)!}\left(\frac{1}{2}\right)^{k+1} + \int_{1/2}^{1} \frac{f^{(n)}(x)}{n!}(1-x)^n dx,$$

and with $t = 0$,

$$\int_0^{1/2} f(x)\,dx = -\sum_{k=1}^{n-1} \frac{f^{(k)}(1/2)}{(k+1)!}\left(-\frac{1}{2}\right)^{k+1} + \int_{0}^{1/2} \frac{f^{(n)}(x)}{n!}(-x)^n dx.$$

When we add these, the terms for odd k cancel, while the terms for even k vanish by hypothesis. It follows that

$$\int_0^1 f(x)\,dx = \int_0^1 g(x) f^{(n)}(x)\,dx,$$

where

$$g(x) = \begin{cases} (-x)^n/n! & \text{when } 0 \le x \le 1/2; \\ (1-x)^n/n! & \text{when } 1/2 \le x \le 1. \end{cases}$$

Now the desired inequality follows from the Cauchy–Schwarz inequality, because

$$\int_0^1 g(x)^2\,dx = \int_0^{1/2} \frac{x^{2n}}{(n!)^2}\,dx + \int_{1/2}^1 \frac{(1-x)^{2n}}{(n!)^2}\,dx = \frac{1}{(2n+1)2^{2n}(n!)^2}.$$

Also solved by U. Abel (Germany), K. F. Andersen (Germany), P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), R. Dutta (India), N. Grivaux (France), A. Harnist (France), E. A. Herman, K. Koo (China), O. Kouba (Syria), M. E. Kuczma (Poland), J. H. Lindsey II, O. P. Lossers (Netherlands), F. Marino (Italy), V. Mikayelyan (Armenia), R. Nandan, M. Omarjee (France), Á. Plaza & F. Perdomo (Spain), M. A. Prasad (India), M. Sawhney, A. Stadler (Switzerland), R. Stong, R. Tauraso (Italy), E. I. Verriest, T. Wiandt, L. Zhou, GCHQ Problem Solving Group (U. K.), Missouri State University Problem Solving Group, and the proposer.