A Lower Bound for a Product of Integrals

11933 [2016, 832]. *Proposed by José M. Pacheco and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.* For positive integer n, let $H_n = \sum_{k=1}^{n} 1/k$. Prove

$$\int_0^1 \frac{1}{x+1}\,dx \cdot \int_0^1 \frac{x+1}{x^2+x+1}\,dx \cdots \int_0^1 \frac{x^{n-2}+\cdots+x+1}{x^{n-1}+\cdots+x+1}\,dx \ge \frac{1}{H_n}.$$

Solution by Travis D. Cunningham, Bellamy Creek Correctional Facility, Ionia, MI. Put $f_k(x) = x^k + \cdots + x + 1$, noting that $f_k(x) > 0$ on $[0, 1]$. By the generalized Hölder inequality, we obtain

$$\begin{aligned} 1 &= \left(\int_0^1 \left(\frac{1}{f_1(x)} \cdot \frac{f_1(x)}{f_2(x)} \cdots \frac{f_{n-2}(x)}{f_{n-1}(x)} \cdot f_{n-1}(x)\right)^{1/n} dx\right)^n \\ &\le \int_0^1 \frac{1}{f_1(x)}\,dx \cdot \int_0^1 \frac{f_1(x)}{f_2(x)}\,dx \cdots \int_0^1 \frac{f_{n-2}(x)}{f_{n-1}(x)}\,dx \cdot \int_0^1 f_{n-1}(x)\,dx. \end{aligned}$$

Since

$$\int_0^1 f_{n-1}(x)\,dx = H_n,$$

this yields the desired inequality.

Solution II by Nicholas C. Singer, Annandale, VA. It is easily seen by induction that $H_{k+1} \le 1 + k/2$. Let $f_k(x) = 1 + x + \cdots + x^k$ as in the previous solution. For $k \ge 1$, apply the arithmetic-mean–geometric-mean inequality to $1, x, x^2, \ldots, x^k$ to obtain

$$\frac{f_k(x)}{k+1} \ge (1 \cdot x \cdot x^2 \cdots x^k)^{1/(k+1)} = x^{k/2}.$$

Thus,

$$\frac{x^k}{f_k(x)} \le \frac{x^{k/2}}{k+1},$$

and hence

$$\int_0^1 \frac{x^k}{f_k(x)}\,dx \le \frac{1}{(k+1)(k/2+1)} \le \frac{1/(k+1)}{H_{k+1}}.$$

Therefore,

$$\int_0^1 \frac{f_{k-1}(x)}{f_k(x)}\,dx = \int_0^1 \left(1 - \frac{x^k}{f_k(x)}\right) dx \ge 1 - \frac{1/(k+1)}{H_{k+1}} = \frac{H_k}{H_{k+1}},$$

and so

$$\int_0^1 \frac{1}{x+1}\,dx \cdot \int_0^1 \frac{x+1}{x^2+x+1}\,dx \cdots \int_0^1 \frac{x^{n-2}+\cdots+x+1}{x^{n-1}+\cdots+x+1}\,dx$$
$$\ge \frac{H_1}{H_2}\cdot\frac{H_2}{H_3}\cdots\frac{H_{n-1}}{H_n} = \frac{1}{H_n}.$$

Editorial comment. Several solvers pointed out that the left side of the requested inequality has a positive lower bound. Indeed, Singer pointed out that his solution proves a stronger inequality, and this inequality gives such a lower bound:

$$\prod_{k=1}^{n-1}\int_0^1 \frac{f_{k-1}(x)}{f_k(x)}\,dx \ge \prod_{k=1}^{n-1}\left(1 - \frac{1}{(k+1)(k/2+1)}\right)$$
$$= \prod_{k=1}^{n-1}\frac{k(k+3)}{(k+1)(k+2)} = \frac{n+2}{3n} > \frac{1}{3}.$$

Since $H_n > 3$ for $n \ge 11$, this implies that it suffices to verify the desired inequality for $n \le 10$.

Also solved by P. Acosta, A. Ali (India), R. Bagby, R. Boukharfane (France), P. Bracken, R. Chapman (U. K.), H. Chen, M. Bello & M. Benito & Ó. Ciaurri & E. Fernéndez & L. Roncal (Spain), P. P. Dályay (Hungary), R. Dutta (India), N. Grivaux (France), J. Grzesik, A. Harnist & M. Cook (France), E. Herman, E. Ionascu, B. Karaivanov (U. S. A.) & T. S. Vassilev (Canada), K. Koo (China), O. Kouba (Syria), M. Kuczma (Poland), K.-W. Lau (China), O. P. Lossers (Netherlands), R. Nandan, M. Omarjee (France) M. A. Prasad (India), E. Schmeichel, A. Stenger, R. Stong, R. Tauraso (Italy), D. Tyler, R. Z. Voepel, M. Vowe (Switzerland), R. Wiandt, J. Zacharias, Y. Zhao, L. Zhou, GCHQ Problem Solving Group (U. K.), NSA Problems Group, and the proposer.