Inequality of Integrals

11884 [2016, 98]. *Proposed by Cezar Lupu, University of Pittsburgh, Pittsburgh, PA and Tudorel Lupu, Decebal High School, Constanţa, Romania.* Let f be a real-valued function on [0, 1] such that f and its first two derivatives are continuous. Prove that if $f(1/2) = 0$, then

$$\int_0^1 (f''(x))^2\,dx \geq 320 \left(\int_0^1 f(x)\,dx\right)^2.$$

Solution by Henry Ricardo, Tappan, NY. If we let $g(x) = f(x) + f(1-x)$, then $g''(x)$ is continuous and $g(1/2) = g'(1/2) = 0$. Integrate $g(x)$ twice by parts to get

$$\int_0^{1/2} g(x)\,dx = -\int_0^{1/2} xg'(x)\,dx = \frac{1}{2}\int_0^{1/2} x^2 g''(x)\,dx.$$

The Cauchy–Schwarz inequality implies

$$\left(\int_0^{1/2} g(x)\,dx\right)^2 \leq \frac{1}{4}\int_0^{1/2} x^4\,dx \cdot \int_0^{1/2} (g''(x))^2\,dx = \frac{1}{640}\int_0^{1/2} (g''(x))^2\,dx. \qquad (1)$$

Using the inequality $(a+b)^2 \leq 2(a^2+b^2)$, we deduce

$$g''(x)^2 = (f''(x) + f''(1-x))^2 \leq 2((f''(x))^2 + (f''(1-x))^2).$$

Hence,

$$\int_0^{1/2} (g''(x))^2\,dx \leq 2\int_0^{1/2} ((f''(x))^2 + (f''(1-x))^2)\,dx = 2\int_0^1 (f''(x))^2\,dx. \qquad (2)$$

From (1) and (2),

$$\int_0^1 \left(f''(x)\right)^2 dx \geq 320\left(\int_0^{1/2} g(x)\,dx\right)^2 = 320\left(\int_0^{1/2} (f(x) + f(1-x))\,dx\right)^2$$
$$= 320\left(\int_0^1 f(x)\,dx\right)^2.$$

Also solved by A. Ali (India), T. Amdeberhan, K. F. Andersen (Canada), M. Bello & M. Benito & Ó. Ciaurri & E. Fernández & L. Roncal (Spain), P. Bracken, R. Chapman (U. K.), H. Chen, P. P. Dályay (Hungary), R. Dutta (India), N. Grivaux (France), A. Harnist & M. Cook (France), E. A. Herman, R. Howard, O. Kouba (Syria), K.-W. Lau (China), J. H. Lindsey II, P. W. Lindstrom, O. P. Lossers (Netherlands), V. Mikayelyan