A Hybrid High-Order method for creeping flows of non-Newtonian fluids

Journée des doctorant·e·s de l'IMAG – 12 février 2020

André HARNIST
ACSIOM team
Supervised by Daniele Di Pietro
In collaboration with Michele Botti and Daniel Castanon Quiroz

Outline

Outline

Newtonian and non-Newtonian fluids

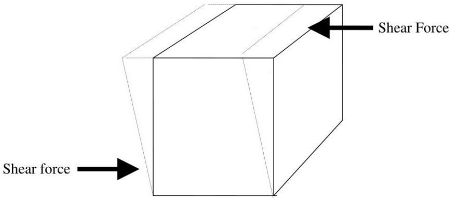
We can distinguish 3 categories of fluids with respect to their viscosity:

- Newtonian: viscosity is constant with increased stress.
  examples: air, water
- non-Newtonian:
  - pseudoplastic: viscosity decreases with increased stress.
    examples: blood, ketchup
  - dilatant: viscosity increases with increased stress.
    examples: oobleck, quicksand

Strain rate-shear stress function

We characterize the movement of a fluid with two properties:

- Strain rate: the change in strain (deformation) of the fluid with respect to time.
- Shear stress: the component of stress coplanar with the fluid cross section.

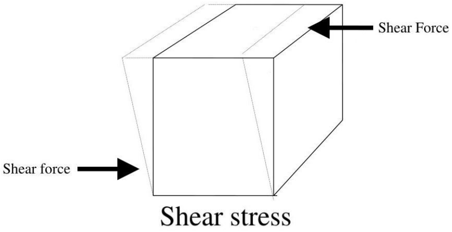


Shear stress

Strain rate-shear stress function

We characterize the movement of a fluid with two properties:

- Strain rate: the change in strain (deformation) of the fluid with respect to time.
- Shear stress: the component of stress coplanar with the fluid cross section.



These properties are quantified by a strain rate-shear stress function

$$\boldsymbol{\sigma} : \mathbb{R}_{\mathrm{s}}^{d\times d} \to \mathbb{R}_{\mathrm{s}}^{d\times d},$$

where $\mathbb{R}_{\mathrm{s}}^{d\times d} := \{\boldsymbol{\tau} \in \mathbb{R}^{d\times d} : \boldsymbol{\tau}^{\mathrm{T}} = \boldsymbol{\tau}\}$, $d \in \mathbb{N}^*$.

Newtonian fluid

- A Newtonian fluid is one for which $\boldsymbol{\sigma}$ is linear: For all $\boldsymbol{\tau} \in \mathbb{R}_{\mathrm{s}}^{d \times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = \mu \boldsymbol{\tau},$$

  where $\mu \in \mathbb{R}^{+}$ is the shear viscosity of the fluid.

- A non-Newtonian fluid is a fluid which is not Newtonian. There are several laws characterizing it:
  - Power-law
  - Carreau–Yasuda
  - Yeleswarapu
  - Quemada
  - Cross
  - ...

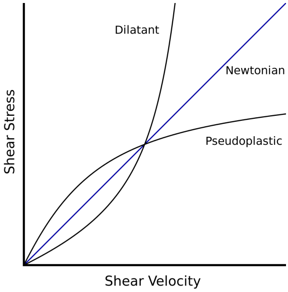
Power-law fluid

A power-law fluid is one for which $\boldsymbol{\sigma}$ is such that for all $\boldsymbol{\tau} \in \mathbb{R}_{\mathrm{s}}^{d\times d}$,

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = \mu|\boldsymbol{\tau}|_{d\times d}^{r-2}\boldsymbol{\tau},$$

where $\mu > 0$ is the flow consistency index and $r > 1$ is the flow behavior index.

- If $r < 2$, the fluid is pseudoplastic (e.g. blood, ketchup).
- If $r = 2$, the fluid is Newtonian (e.g. air, water).
- If $r > 2$, the fluid is dilatant (e.g. oobleck, quicksand).



Outline

The steady Stokes problem

Let $\Omega \subset \mathbb{R}^d$ denote a bounded, connected polyhedral open set with Lipschitz boundary $\partial\Omega$.
The steady Stokes problem reads: Find $\boldsymbol{u}$ and p such that

$$
\begin{aligned}
-\boldsymbol{\nabla}\cdot(\mu\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) + \boldsymbol{\nabla}p &= \boldsymbol{f} \quad \text{in } \Omega,\\
\boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \partial\Omega,\\
\int_\Omega p &= 0,
\end{aligned}
$$

where

- $\boldsymbol{f} : \Omega \to \mathbb{R}^d$ is the force applied on the fluid,
- $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is the velocity of the fluid,
- $p : \Omega \to \mathbb{R}$ is the pressure of the fluid,
- $\mu \in (0, +\infty)$ is the viscosity of the fluid.

The steady generalized Stokes problem

Let $\Omega \subset \mathbb{R}^d$ denote a bounded, connected polyhedral open set with Lipschitz boundary $\partial\Omega$.
The steady generalized Stokes problem reads: Find $\boldsymbol{u}$ and p such that

$$
\begin{aligned}
-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\cdot,\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) + \boldsymbol{\nabla}p &= \boldsymbol{f} && \text{in } \Omega,\\
\boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0 && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega,\\
\int_\Omega p &= 0,
\end{aligned}
$$

where

- $\boldsymbol{f} : \Omega \to \mathbb{R}^d$ is the force applied on the fluid,
- $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ is the velocity of the fluid,
- $p : \Omega \to \mathbb{R}$ is the pressure of the fluid,
- $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_{\mathrm{s}}^{d\times d} \to \mathbb{R}_{\mathrm{s}}^{d\times d}$ is the strain rate-shear stress function of the fluid.

Assumptions on $\boldsymbol{\sigma}$ – The Hilbert assumption

Hilbert assumption

- $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_{\mathrm{s}}^{d\times d} \to \mathbb{R}_{\mathrm{s}}^{d\times d}$ is measurable,
- $\boldsymbol{\sigma}(\cdot, \mathbf{0}) \in L^2(\Omega, \mathbb{R}_{\mathrm{s}}^{d\times d})$ a.e. in Ω,
- There is $\sigma_{\mathrm{hc}}, \sigma_{\mathrm{sm}} \in (0, +\infty)$ such that,

$$|\boldsymbol{\sigma}(\cdot, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\cdot, \boldsymbol{\eta})|_{d\times d} \leqslant \sigma_{\mathrm{hc}} |\boldsymbol{\tau} - \boldsymbol{\eta}|_{d\times d},$$
$$(\boldsymbol{\sigma}(\cdot, \boldsymbol{\tau}) - \boldsymbol{\sigma}(\cdot, \boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \geqslant \sigma_{\mathrm{sm}} |\boldsymbol{\tau} - \boldsymbol{\eta}|_{d\times d}^2,$$

for all $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}_{\mathrm{s}}^{d\times d}$, a.e. in Ω.

$\rightsquigarrow$ see [Botti, Di Pietro, Sochala; 2017]

Assumptions on $\boldsymbol{\sigma}$ – The Sobolev assumption

Sobolev assumption

There is $r \in (1, +\infty)$ such that:

- $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{s}} \to \mathbb{R}^{d\times d}_{\mathrm{s}}$ is measurable,
- $\boldsymbol{\sigma}(\cdot, \mathbf{0}) \in L^{r'}(\Omega, \mathbb{R}^{d\times d}_{\mathrm{s}})$ a.e. in Ω with $r' := \frac{r}{r-1}$,
- There is $\sigma_{\mathrm{hc}}, \sigma_{\mathrm{sm}} \in (0, +\infty)$ such that,

$$|\boldsymbol{\sigma}(\cdot,\boldsymbol{\tau}) - \boldsymbol{\sigma}(\cdot,\boldsymbol{\eta})|_{d\times d} \leqslant \begin{cases} \sigma_{\mathrm{hc}} |\boldsymbol{\tau} - \boldsymbol{\eta}|^{r-1}_{d\times d} & \text{if } r < 2 \\ \sigma_{\mathrm{hc}} |\boldsymbol{\tau} - \boldsymbol{\eta}|_{d\times d} (|\boldsymbol{\tau}|^{r-2}_{d\times d} + |\boldsymbol{\eta}|^{r-2}_{d\times d}) & \text{if } r \geqslant 2 \end{cases},$$

$$(\boldsymbol{\sigma}(\cdot,\boldsymbol{\tau}) - \boldsymbol{\sigma}(\cdot,\boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \geqslant \begin{cases} \sigma_{\mathrm{sm}} |\boldsymbol{\tau} - \boldsymbol{\eta}|^{2} (|\boldsymbol{\tau}|_{d\times d} + |\boldsymbol{\eta}|_{d\times d})^{r-2} & \text{if } r < 2 \\ \sigma_{\mathrm{sm}} |\boldsymbol{\tau} - \boldsymbol{\eta}|^{r}_{d\times d} & \text{if } r \geqslant 2 \end{cases},$$

for all $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\mathrm{s}}$, a.e. in Ω.

$\rightsquigarrow$ see [Di Pietro, Droniou; 2017]

Assumptions on $\boldsymbol{\sigma}$ – The singular Sobolev assumption

Singular Sobolev assumption

There is $r \in (1, +\infty)$ such that:

- $\boldsymbol{\sigma} : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{s}} \to \mathbb{R}^{d\times d}_{\mathrm{s}}$ is measurable,
- $\boldsymbol{\sigma}(\cdot, \mathbf{0}) \in L^{r'}(\Omega, \mathbb{R}^{d\times d}_{\mathrm{s}})$ a.e. in Ω with $r' := \frac{r}{r-1}$,
- There is $\sigma_{\mathrm{de}} \in [0, +\infty)$ and $\sigma_{\mathrm{hc}}, \sigma_{\mathrm{sm}} \in (0, +\infty)$ such that,

$$|\boldsymbol{\sigma}(\cdot,\boldsymbol{\tau}) - \boldsymbol{\sigma}(\cdot,\boldsymbol{\eta})|_{d\times d} \leqslant \sigma_{\mathrm{hc}} \left({\sigma_{\mathrm{de}}}^r + |\boldsymbol{\tau}|^r_{d\times d} + |\boldsymbol{\eta}|^r_{d\times d}\right)^{\frac{r-r^\circ}{r}} |\boldsymbol{\tau} - \boldsymbol{\eta}|^{r^\circ - 1}_{d\times d},$$

$$(\boldsymbol{\sigma}(\cdot,\boldsymbol{\tau}) - \boldsymbol{\sigma}(\cdot,\boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \geqslant \sigma_{\mathrm{sm}} \left({\sigma_{\mathrm{de}}}^r + |\boldsymbol{\tau}|^r_{d\times d} + |\boldsymbol{\eta}|^r_{d\times d}\right)^{\frac{r^\circ-2}{r}} |\boldsymbol{\tau} - \boldsymbol{\eta}|^{r+2-r^\circ}_{d\times d},$$

for all $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}^{d\times d}_{\mathrm{s}}$, a.e. in Ω, where

$$r^\circ := \min(r, 2)$$

is the singular exponent of r.

Weak formulation

We assume $\boldsymbol{f} \in L^{r'}(\Omega, \mathbb{R}^d)$. We define

- $\boldsymbol{U} \coloneqq W_0^{1,r}(\Omega, \mathbb{R}^d) = \{\boldsymbol{v} \in W^{1,r}(\Omega, \mathbb{R}^d) \ : \ \boldsymbol{v}_{|\partial\Omega} = \boldsymbol{0}\}$
- $P \coloneqq L_0^{r'}(\Omega, \mathbb{R}) = \left\{q \in L^{r'}(\Omega, \mathbb{R}) \ : \ \int_\Omega q = 0\right\}$

The weak formulation of the steady generalized Stokes problem reads:
Find $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ such that

$$\begin{aligned} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \qquad && \forall \boldsymbol{v} \in \boldsymbol{U}, \\ -b(\boldsymbol{u}, q) &= 0 && \forall q \in P, \end{aligned}$$

where, for all $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{U}$ and all $q \in P$,

$$a(\boldsymbol{w}, \boldsymbol{v}) \coloneqq \int_\Omega \boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{w}) : \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v},$$

$$b(\boldsymbol{v}, q) \coloneqq -\int_\Omega (\boldsymbol{\nabla} \cdot \boldsymbol{v}) q.$$

Outline

Hybrid High–Order

Hybrid: there are two kinds of unknowns located on the mesh and its skeleton.

High-Order: the unknowns live in broken polynomial spaces of degree $k \in \mathbb{N}$.

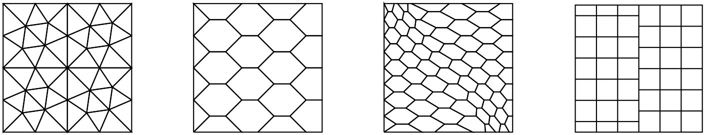
At the **core of HHO methods** are:

- Local reconstructions of relevant differential operators mimicking IBP.
- Stabilization functions satisfying suitable properties.

This approach possesses **several attractive features**.

General features

- Arbitrary approximation order ($k \geqslant 0$)
- Formulation valid for arbitrary space dimension.
- Seamless treatment of nonconforming mesh refinement.



- Moderate computational costs thanks to static condensation.
- Inf-sup stable discretizations.

Minimal bibliography

- A Hybrid High-Order method for nonlinear elasticity

  [Michele Botti, Daniele Di Pietro and Pierre Sochala, 2017]

- A Hybrid High-Order method for Leray-Lions elliptic equations on general meshes

  [Daniele A. Di Pietro and Jérôme Droniou, 2016]

- A Hybrid High-Order method for the steady incompressible Navier–Stokes problem

  [Daniele Di Pietro and Stella Krell, 2018]

- $W^{s,p}$-approximation properties of elliptic projectors on polynomial spaces, with application to the error analysis of a Hybrid High-Order discretisation of Leray–Lions problems

  [Daniele Di Pietro and Jérôme Droniou, 2017]

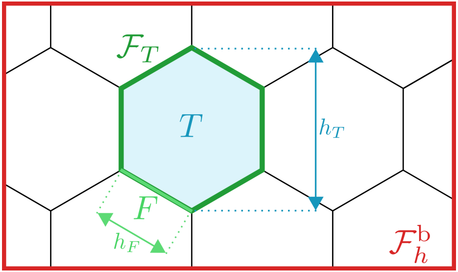
Mesh and notations

Let $h \in (0, +\infty)$. We define a mesh of Ω as a couple $(\mathcal{T}_h, \mathcal{F}_h)$ such that

- $\mathcal{T}_h$ is a finite collection of polyhedral elements T with diameter h_T,
- $\mathcal{F}_h$ is a finite collection of planar faces F with diameter h_F,
- $\bigcup_{T\in\mathcal{T}_h} \overline{T} = \overline{\Omega}$ and $\max_{T\in\mathcal{T}_h} h_T = h$,
- $(\mathcal{T}_h, \mathcal{F}_h)$ satisfies some geometrical requirements...

We also define the following subsets of $\mathcal{F}_h$:

- $\mathcal{F}_h^{\mathrm{b}} \coloneqq \{F \in \mathcal{F}_h \ : \ F \subset \partial\Omega\}$,
- $\mathcal{F}_T \coloneqq \{F \in \mathcal{F}_h \ : \ F \subset \partial T\}$ for all $T \in \mathcal{T}_h$.



Discrete spaces and notations

Let $k \geqslant 1$ be the polynomial degree of the HHO method.

- For all $T \in \mathcal{T}_h$, we define the discrete local space:

$$\underline{\boldsymbol{U}}_T^k \coloneqq \mathbb{P}^k(T,\mathbb{R}^d) \times \left(\prod_{F \in \mathcal{F}_T} \mathbb{P}^k(F,\mathbb{R}^d) \right).$$

  We use the discrete notation $\underline{\boldsymbol{v}}_T \coloneqq (\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F \in \mathcal{F}_T}) \in \underline{\boldsymbol{U}}_T^k$.

- We define the discrete global space:

$$\underline{\boldsymbol{U}}_h^k \coloneqq \prod_{T \in \mathcal{T}_h} \underline{\boldsymbol{U}}_T^k.$$

  We use the discrete notation $\underline{\boldsymbol{v}}_h \coloneqq (\underline{\boldsymbol{v}}_T)_{T \in \mathcal{T}_h} \in \underline{\boldsymbol{U}}_h^k$.

Spaces and norms of discrete unknowns

- The discrete space containing velocity unknowns is defined by

$$\underline{\boldsymbol{U}}_{h,0}^k \coloneqq \left\{ \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \; : \; \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \in \mathcal{F}_h^{\rm b} \right\}.$$

We endow $\underline{\boldsymbol{U}}_{h,0}^k$ with the semi-norm $\|\cdot\|_{\varepsilon,r,h}$ defined by

$$\|\underline{\boldsymbol{v}}_h\|_{\varepsilon,r,h}^r \coloneqq \sum_{T \in \mathcal{T}_h} \left(\|\boldsymbol{\nabla}_{\rm s} \boldsymbol{v}_T\|_{L^r(T,\mathbb{R}^{d\times d})}^r + \sum_{F \in \mathcal{F}_T} h_F^{1-r} \|\boldsymbol{v}_F - \boldsymbol{v}_T\|_{L^r(F,\mathbb{R}^d)}^r \right).$$

- The discrete space containing pressure unknowns is defined by

$$P_h^k \coloneqq \left\{ q_h \in L_0^{r'}(\Omega,\mathbb{R}) \; : \; (q_h)_{|T} \in \mathbb{P}^k(T,\mathbb{R}) \quad \forall T \in \mathcal{T}_h \right\}.$$

We endow P_h^k with the norm $\|\cdot\|_{L^{r'}(\Omega,\mathbb{R})}$.

Korn and discrete Korn inequalities

The regularity of Ω yields the Korn inequality: For all $\boldsymbol{v} \in W_0^{1,r}(\Omega, \mathbb{R}^d)$,

$$\|\boldsymbol{v}\|_{W^{1,r}(\Omega,\mathbb{R}^d)} \lesssim \|\boldsymbol{\nabla}_{\rm s}\boldsymbol{v}\|_{L^r(\Omega,\mathbb{R}^{d\times d})}.$$

Theorem. *(discrete Korn inequality)*

We have, for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k$,

$$\|\boldsymbol{v}_h\|_{L^r(\Omega,\mathbb{R}^d)}^r + |\boldsymbol{v}_h|_{W^{1,r}(\mathcal{T}_h,\mathbb{R}^d)}^r \lesssim \|\underline{\boldsymbol{v}}_h\|_{\varepsilon,r,h}^r.$$

As a consequence, $\|\cdot\|_{\varepsilon,r,h}$ is a norm on $\underline{\boldsymbol{U}}_{h,0}^k$.

Hilbertian case $r = 2 \rightsquigarrow$ see [Botti, Di Pietro, Guglielmana; 2019]

Discrete operators

For all $T \in \mathcal{T}_h$, we define:

- the discrete local symmetric gradient $\mathbf{G}_{\mathrm{s},T}^k : \underline{\boldsymbol{U}}_T^k \longrightarrow \mathbb{P}^k(T, \mathbb{R}_{\mathrm{s}}^{d\times d})$ such that, for all $\underline{\boldsymbol{v}}_T \in \underline{\boldsymbol{U}}_T^k$,

$$\int_T \mathbf{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = \int_T \boldsymbol{\nabla}_{\mathrm{s}} \boldsymbol{v}_T : \boldsymbol{\tau} + \sum_{F\in\mathcal{F}_T} \int_F (\boldsymbol{v}_F - \boldsymbol{v}_T)\cdot(\boldsymbol{\tau}\boldsymbol{n}_{TF}) \quad \forall \boldsymbol{\tau} \in \mathbb{P}^k(T, \mathbb{R}_{\mathrm{s}}^{d\times d}).$$

- the discrete local divergence $\mathrm{D}_T^k : \underline{\boldsymbol{U}}_T^k \longrightarrow \mathbb{P}^k(T, \mathbb{R})$ as the trace of the discrete gradient operator: $\mathrm{D}_T^k = \mathrm{tr}(\mathbf{G}_{\mathrm{s},T}^k)$.

There are global versions of these operators: for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$,

$$(\mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{v}}_h)_{|T} := \mathbf{G}_{\mathrm{s},T}^k \underline{\boldsymbol{v}}_T \qquad \forall T \in \mathcal{T}_h,$$

$$(\mathrm{D}_h^k \underline{\boldsymbol{v}}_h)_{|T} := \mathrm{D}_T^k \underline{\boldsymbol{v}}_T \qquad \forall T \in \mathcal{T}_h.$$

Discrete weak formulation

The discrete weak formulation reads: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ such that,

$$\begin{aligned} \mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, p_h) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \qquad && \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \\ -\mathrm{b}_h(\underline{\boldsymbol{u}}_h, q_h) &= 0 && \forall q_h \in P_h^k, \end{aligned}$$

where, for all $\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$ and $q_h \in P_h^k$,

- $\mathrm{a}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) \coloneqq \int_\Omega \boldsymbol{\sigma}(\cdot, \mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{w}}_h) : \mathbf{G}_{\mathrm{s},h}^k \underline{\boldsymbol{v}}_h + \mathrm{s}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h)$
  with s_h a stabilization function mimicking the properties of $\boldsymbol{\sigma}$,
- $\mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h) \coloneqq -\int_\Omega \mathrm{D}_h^k \underline{\boldsymbol{v}}_h \ q_h.$

Properties for a_h and b_h

- For all $\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h, \underline{\boldsymbol{w}}_h \in \underline{\boldsymbol{U}}_h^k$, setting $\underline{\boldsymbol{e}}_h \coloneqq \underline{\boldsymbol{u}}_h - \underline{\boldsymbol{w}}_h$,

$$|\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) - \mathrm{a}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h)| \lesssim \sigma_{\mathrm{hc}} \left(\sigma_{\mathrm{de}}^r + \|\underline{\boldsymbol{u}}_h\|_{\varepsilon,r,h}^r + \|\underline{\boldsymbol{w}}_h\|_{\varepsilon,r,h}^r\right)^{\frac{r-r^\circ}{r}} \|\underline{\boldsymbol{e}}_h\|_{\varepsilon,r,h}^{r^\circ-1} \|\underline{\boldsymbol{v}}_h\|_{\varepsilon,r,h},$$

$$\mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{e}}_h) - \mathrm{a}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{e}}_h) \gtrsim \sigma_{\mathrm{sm}} \left(\sigma_{\mathrm{de}}^r + \|\underline{\boldsymbol{u}}_h\|_{\varepsilon,r,h}^r + \|\underline{\boldsymbol{w}}_h\|_{\varepsilon,r,h}^r\right)^{\frac{2-r^\circ}{r}} \|\underline{\boldsymbol{e}}_h\|_{\varepsilon,r,h}^{r+2-r^\circ}.$$

- For all $q_h \in P_h^k$,

$$\|q_h\|_{L^{r'}(\Omega,\mathbb{R})} \lesssim \sup_{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \|\underline{\boldsymbol{v}}_h\|_{\varepsilon,r,h}=1} \mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h).$$

Well-posedness and a priori bounds

Theorem. *(Well-posedness and a priori bounds)*

There exists a unique solution $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ to the discrete problem. Additionally, the following a priori bounds hold:

$$\|\underline{\boldsymbol{u}}_h\|_{\varepsilon,r,h} \lesssim \left(\sigma_{\rm sm}^{-1} \|\boldsymbol{f}\|_{L^{r'}(\Omega,\mathbb{R}^d)}\right)^{\frac{1}{r-1}} + \left(\sigma_{\rm de}^{2-r^\circ} \sigma_{\rm sm}^{-1} \|\boldsymbol{f}\|_{L^{r'}(\Omega,\mathbb{R}^d)}\right)^{\frac{1}{r+1-r^\circ}}$$

$$\|p_h\|_{L^{r'}(\Omega,\mathbb{R})} \lesssim \sigma_{\rm hc} \left(\sigma_{\rm sm}^{-1} \|\boldsymbol{f}\|_{L^{r'}(\Omega,\mathbb{R}^d)} + \sigma_{\rm de}^{|r-2|(r^\circ-1)} \left(\sigma_{\rm sm}^{-1} \|\boldsymbol{f}\|_{L^{r'}(\Omega,\mathbb{R}^d)}\right)^{\frac{r^\circ-1}{r+1-r^\circ}}\right)$$

Error estimate

Theorem. *(Error estimate)*

Let $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ and $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ solve the continuous and discrete weak formulations, respectively. Assume the additional regularity

- $\boldsymbol{u} \in W^{k+2,r}(\mathcal{T}_h, \mathbb{R}^d)$,
- $\boldsymbol{\sigma}(\cdot, \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) \in W^{1,r'}(\Omega, \mathbb{R}_{\mathrm{s}}^{d\times d}) \cap W^{k+1,r'}(\mathcal{T}_h, \mathbb{R}_{\mathrm{s}}^{d\times d})$,
- $p \in W^{1,r'}(\Omega, \mathbb{R}) \cap W^{k+1,r'}(\mathcal{T}_h, \mathbb{R})$.

Then,

$$\|\underline{\boldsymbol{u}}_h - \underline{\mathbf{I}}_h^k \boldsymbol{u}\|_{\varepsilon,r,h} \lesssim C_1 h^{\frac{k+1}{r+1-r^\circ}} + C_2 h^{\frac{(k+1)(r^\circ-1)}{r+1-r^\circ}},$$

$$\|p_h - \pi_h^k p\|_{L^{r'}(\Omega,\mathbb{R})} \lesssim C_3 h^{k+1} + C_4 h^{(k+1)(r^\circ-1)} + C_5 h^{\frac{(k+1)(r^\circ-1)}{r+1-r^\circ}} + C_6 h^{\frac{(k+1)(r^\circ-1)^2}{r+1-r^\circ}},$$

where $C_1, ..., C_6 \in [0, +\infty)$ depend only on $\boldsymbol{u}, p, \boldsymbol{f}, \sigma_{\mathrm{hc}}, \sigma_{\mathrm{sm}}, \sigma_{\mathrm{de}}, d$, and r.

Numerical results

- For a well chosen $\boldsymbol{f}$, the exact velocity $\boldsymbol{u}$ and pressure p are given by

$$\boldsymbol{u}(x,y) = \frac{a^{-(n+1)}}{n^2(1+b)}\left(\hat{y}^{n-1}\sin(\hat{x}^n)\cos(\hat{y}^n), -\hat{x}^{n-1}\cos(\hat{x}^n)\sin(\hat{y}^n)\right),$$

$$p(x,y) = \frac{a^{1-2n}}{n(1+b)^2}(\hat{x}\hat{y})^{n-1}\cos(a\hat{x}^n)\sin(a\hat{y}^n),$$

  where $a = 10\pi$, $b = (2^{\frac{1}{4}} - 1)^{-1}$, $n = 4$, and $\hat{\cdot} \coloneqq a\frac{\cdot+b}{1+b}$.
- We consider $\Omega = (0,1)^2$ and a distored triangular mesh family.

Numerical results

$$\mathcal{O}^k_{\text{vel}} \coloneqq \frac{(k+1)(r^\circ - 1)}{r + 1 - r^\circ}, \qquad \mathcal{O}^k_{\text{pre}} \coloneqq \frac{(k+1)(r^\circ - 1)^2}{r + 1 - r^\circ}$$

$r = 1.5$ and $k = 1$				
ndofs	$\Vert \underline{\boldsymbol{u}}_h - \underline{\mathbf{I}}_h^k \boldsymbol{u} \Vert_{\varepsilon,r,h}$	$\text{ocv}_{\boldsymbol{u}}$	$\Vert p_h - \pi_h^k p \Vert_{L^{r'}(\Omega,\mathbb{R})}$	ocv_p
268	$4.94 \cdot 10^{-5}$	1.00	$4.83 \cdot 10^{-4}$	0.50
1148	$1.80 \cdot 10^{-5}$	1.46	$1.65 \cdot 10^{-4}$	1.55
4732	$5.13 \cdot 10^{-6}$	1.81	$4.87 \cdot 10^{-5}$	1.76
18548	$1.52 \cdot 10^{-6}$	1.75	$1.53 \cdot 10^{-5}$	1.67
76096	$4.24 \cdot 10^{-7}$	1.85	$3.31 \cdot 10^{-6}$	2.21
298676	$1.21 \cdot 10^{-7}$	1.81	$1.14 \cdot 10^{-6}$	1.53

Numerical results

$$\mathcal{O}_{\text{vel}}^k \coloneqq \frac{(k+1)(r^\circ - 1)}{r + 1 - r^\circ}, \qquad \mathcal{O}_{\text{pre}}^k \coloneqq \frac{(k+1)(r^\circ - 1)^2}{r + 1 - r^\circ}$$

$r = 1.5$ and $k = 2$				
ndofs	$\lVert \underline{\boldsymbol{u}}_h - \underline{\mathbf{I}}_h^k \boldsymbol{u} \rVert_{\varepsilon,r,h}$	$\text{ocv}_{\boldsymbol{u}}$	$\lVert p_h - \pi_h^k p \rVert_{L^{r'}(\Omega,\mathbb{R})}$	ocv_p
402	$2.77 \cdot 10^{-5}$	1.50	$3.68 \cdot 10^{-4}$	0.75
1722	$8.78 \cdot 10^{-6}$	1.66	$9.87 \cdot 10^{-5}$	1.90
7098	$2.18 \cdot 10^{-6}$	2.01	$2.23 \cdot 10^{-5}$	2.14
27822	$7.84 \cdot 10^{-7}$	1.48	$9.01 \cdot 10^{-6}$	1.31
114144	$1.81 \cdot 10^{-7}$	2.12	$2.08 \cdot 10^{-6}$	2.11
448014	$4.67 \cdot 10^{-8}$	1.95	$7.79 \cdot 10^{-7}$	1.42

Numerical results

$$\mathcal{O}^k_{\text{vel}} := \frac{(k+1)(r^\circ - 1)}{r+1-r^\circ}, \qquad \mathcal{O}^k_{\text{pre}} := \frac{(k+1)(r^\circ - 1)^2}{r+1-r^\circ}$$

$r = 2$ and $k = 1$				
ndofs	$\Vert \underline{\boldsymbol{u}}_h - \underline{\mathbf{I}}_h^k \boldsymbol{u} \Vert_{\varepsilon,r,h}$	$\mathrm{ocv}_{\boldsymbol{u}}$	$\Vert p_h - \pi_h^k p \Vert_{L^{r'}(\Omega,\mathbb{R})}$	ocv_p
268	$1.31 \cdot 10^{-4}$	2.00	$2.82 \cdot 10^{-5}$	2.00
1148	$4.88 \cdot 10^{-5}$	1.43	$1.04 \cdot 10^{-5}$	1.44
4732	$1.40 \cdot 10^{-5}$	1.80	$3.06 \cdot 10^{-6}$	1.76
18548	$3.73 \cdot 10^{-6}$	1.91	$8.47 \cdot 10^{-7}$	1.85
76096	$9.38 \cdot 10^{-7}$	1.99	$2.17 \cdot 10^{-7}$	1.97
298676	$2.41 \cdot 10^{-7}$	1.96	$5.67 \cdot 10^{-8}$	1.93

Numerical results

$$\mathcal{O}^k_{\text{vel}} \coloneqq \frac{(k+1)(r^\circ - 1)}{r + 1 - r^\circ}, \qquad \mathcal{O}^k_{\text{pre}} \coloneqq \frac{(k+1)(r^\circ - 1)^2}{r + 1 - r^\circ}$$

$r = 2$ and $k = 2$				
ndofs	$\Vert \underline{\boldsymbol{u}}_h - \underline{\mathbf{I}}^k_h \boldsymbol{u} \Vert_{\varepsilon,r,h}$	$\text{ocv}_{\boldsymbol{u}}$	$\Vert p_h - \pi^k_h p \Vert_{L^{r'}(\Omega,\mathbb{R})}$	ocv_p
402	$6.77 \cdot 10^{-5}$	3.00	$1.36 \cdot 10^{-5}$	3.00
1722	$1.37 \cdot 10^{-5}$	2.30	$2.73 \cdot 10^{-6}$	2.32
7098	$2.03 \cdot 10^{-6}$	2.76	$4.30 \cdot 10^{-7}$	2.67
27822	$2.72 \cdot 10^{-7}$	2.90	$5.88 \cdot 10^{-8}$	2.87
114144	$3.40 \cdot 10^{-8}$	3.00	$7.09 \cdot 10^{-9}$	3.05
448014	$4.38 \cdot 10^{-9}$	2.95	$9.12 \cdot 10^{-10}$	2.96

Numerical results

$$\mathcal{O}^k_{\mathrm{vel}} \coloneqq \frac{(k+1)(r^\circ - 1)}{r+1-r^\circ}, \qquad \mathcal{O}^k_{\mathrm{pre}} \coloneqq \frac{(k+1)(r^\circ - 1)^2}{r+1-r^\circ}$$

$r = 2.5$ and $k = 1$				
ndofs	$\Vert\underline{\boldsymbol{u}}_h - \underline{\mathbf{I}}_h^k \boldsymbol{u}\Vert_{\varepsilon,r,h}$	$\mathrm{ocv}_{\boldsymbol{u}}$	$\Vert p_h - \pi_h^k p\Vert_{L^{r'}(\Omega,\mathbb{R})}$	ocv_p
268	$3.90 \cdot 10^{-3}$	1.33	$1.74 \cdot 10^{-4}$	1.33
1148	$2.22 \cdot 10^{-3}$	0.82	$8.16 \cdot 10^{-5}$	1.09
4732	$1.05 \cdot 10^{-3}$	1.08	$3.07 \cdot 10^{-5}$	1.41
18548	$4.73 \cdot 10^{-4}$	1.15	$1.07 \cdot 10^{-5}$	1.52
76096	$1.88 \cdot 10^{-4}$	1.34	$3.55 \cdot 10^{-6}$	1.60
298676	$6.54 \cdot 10^{-5}$	1.52	$1.20 \cdot 10^{-6}$	1.57

Numerical results

$$\mathcal{O}^k_{\mathrm{vel}} \coloneqq \frac{(k+1)(r^\circ - 1)}{r+1-r^\circ}, \qquad \mathcal{O}^k_{\mathrm{pre}} \coloneqq \frac{(k+1)(r^\circ - 1)^2}{r+1-r^\circ}$$

$r = 2.5$ and $k = 2$				
ndofs	$\Vert \underline{\boldsymbol{u}}_h - \underline{\mathbf{I}}^k_h \boldsymbol{u} \Vert_{\varepsilon,r,h}$	$\mathrm{ocv}_{\boldsymbol{u}}$	$\Vert p_h - \pi^k_h p \Vert_{L^{r'}(\Omega,\mathbb{R})}$	ocv_p
402	$2.76 \cdot 10^{-3}$	2.00	$1.03 \cdot 10^{-4}$	2.00
1722	$1.00 \cdot 10^{-3}$	1.46	$2.57 \cdot 10^{-5}$	2.01
7098	$2.97 \cdot 10^{-4}$	1.76	$5.64 \cdot 10^{-6}$	2.19
27822	$7.74 \cdot 10^{-5}$	1.94	$1.17 \cdot 10^{-6}$	2.27
114144	$1.73 \cdot 10^{-5}$	2.16	$2.18 \cdot 10^{-7}$	2.42
448014	$3.59 \cdot 10^{-6}$	2.27	$4.29 \cdot 10^{-8}$	2.35

Conclusion

No matter what r is chosen, the orders of convergence are in agreement with the theoretical predictions.

Next steps include:

- Convergence by compactness on the Navier-Stokes equations
- Analysis for rheopecty and thixotropic fluids

Thank you for your attention.