Hybrid High-Order Methods for Complex Problems in Fluid Mechanics

D. Castanon Quiroz[1], D. A. Di Pietro[2] and A. Harnist[3]

[1]IIMAS-UNAM, CDMX, Mexico [2,3]IMAG, Univ. Montpellier, CNRS, France [3]Current affiliation: Inria Paris & École des Ponts, France

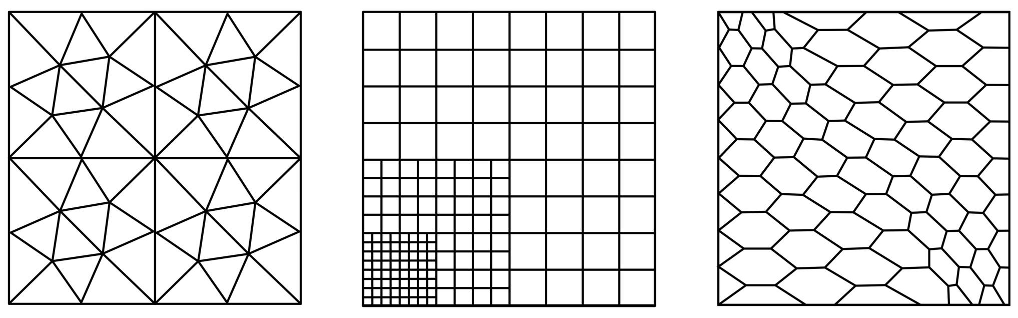
andre.harnist@inria.fr, andreharnist.fr

Introduction

We present the work achieved in [2], where we develop a Hybrid High-Order (HHO) method for a generalized Navier-Stokes problem adapted not only to non-Newtonian fluids, but also fluids with non-classical convective behaviour. It is a generalization of the HHO methods implemented for the generalized Stokes problem in [1] and the Navier-Stokes problem in [6], based on the works of [3, 4] and [5]. The space discretization hinges on local reconstruction operators from hybrid polynomial unknowns at the elements and faces.

The HHO method has several assets:

- handles general polyhedral meshes with seamless treatment of nonconforming mesh refinement;



- dimension-independent implementation;
- arbitrary order (better accuracy for a fixed mesh or fewer elements for a given precision);
- offers stability for inf-sup condition;
- faithfully reproduces non-Newtonian behaviours.

We show a convergence to minimal regularity solutions, and a detailed error estimate in Sobolev-like norms, under some assumptions on the viscosity and convective laws. For the sake of simplicity, we will focus on a Carreau-Yasuda viscosity law and a power-like convective law since they verify the assumptions required to obtain the convergence and error estimate. Finally, we show an application to the lid-driven cavity problem which is a very classical test that, despite incompatible boundary conditions, demonstrates the performance of the method in situations closer to real-life problems.

Generalized Navier–Stokes Problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$, denote a bounded, connected, polyhedral open set with Lipschitz boundary $\partial\Omega$. We consider the incompressible flow of a fluid occupying Ω and subjected to a volumetric force field $\boldsymbol{f} : \Omega \to \mathbb{R}^d$, governed by the following generalized Navier–Stokes problem: Find the velocity field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and the pressure field $p : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}(\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\chi}(\boldsymbol{u}) + \boldsymbol{\nabla} p = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega, \\ \int_\Omega p = 0, & \end{cases}$$

where the viscosity law $\boldsymbol{\sigma} : \mathbb{R}^{d\times d}_{\mathrm{s}} \to \mathbb{R}^{d\times d}_{\mathrm{s}}$ and the convective law $\boldsymbol{\chi} : \mathbb{R}^d \to \mathbb{R}^d$ are defined such that

$$\boldsymbol{\sigma}(\boldsymbol{\tau}) = \kappa(\delta + |\boldsymbol{\tau}|)_{d\times d}^{r-2}\boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in \mathbb{R}^{d\times d}_{\mathrm{s}} \quad \text{and} \quad \boldsymbol{\chi}(\boldsymbol{w}) = \nu|\boldsymbol{w}|^{s-2}\boldsymbol{w} \quad \forall \boldsymbol{w} \in \mathbb{R}^d,$$

with $\kappa, \nu \in (0,\infty)$, $\delta \in [0,\infty)$, and where the viscous index $r \in (1,\infty)$ allows us to consider non-Newtonian fluids, while the convective index $s \in (1,\infty)$ generalizes the classical convective term (with $s = 2$) to power-like convective behaviours.

The Hybrid High-Order Method

Discrete spaces and norms

For a mesh $\mathcal{T}_h$ of size $h \in \mathcal{H} \subset (0,\infty)$ and a polynomial degree $k \geq 1$, we define the discrete global space

$$\underline{\boldsymbol{U}}_h^k := \left\{\underline{\boldsymbol{v}}_h := ((\boldsymbol{v}_T)_{T\in\mathcal{T}_h}, (\boldsymbol{v}_F)_{F\in\mathcal{F}_h}) \;:\; \boldsymbol{v}_T \in \mathbb{P}^k(T)^d \;\; \forall T \in \mathcal{T}_h \;\text{ and }\; \boldsymbol{v}_F \in \mathbb{P}^k(F)^d \;\; \forall F \in \mathcal{F}_h\right\}.$$

For all $T \in \mathcal{T}_h$, we denote by $\underline{\boldsymbol{U}}_T^k$ the restriction $\underline{\boldsymbol{U}}_h^k$ to T, and, for all $\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k$, we let $\underline{\boldsymbol{v}}_T := (\boldsymbol{v}_T, (\boldsymbol{v}_F)_{F\in\mathcal{F}_T}) \in \underline{\boldsymbol{U}}_T^k$, and $(\boldsymbol{v}_h)_{|T} := \boldsymbol{v}_T$. The discrete spaces containing the velocity and pressure unknowns are respectively

$$\underline{\boldsymbol{U}}_{h,0}^k := \left\{\underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_h^k \;:\; \boldsymbol{v}_F = \boldsymbol{0} \quad \forall F \in \mathcal{F}_h^{\mathrm{b}}\right\} \quad \text{and} \quad P_h^k := \left\{q_h \in L_0^{r'}(\Omega) \;:\; (q_h)_{|T} \in \mathbb{P}^k(T) \quad \forall T \in \mathcal{T}_h\right\}.$$

We endow P_h^k with the norm $\|\cdot\|_{L^{r'}(\Omega)}$, and $\underline{\boldsymbol{U}}_{h,0}^k$ with the norm $\|\cdot\|_{\boldsymbol{\varepsilon},r,h}$ defined such that

$$\|\underline{\boldsymbol{v}}_h\|_{\boldsymbol{\varepsilon},r,h}^r := \sum_{T\in\mathcal{T}_h}\left(\|\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{v}_T\|_{L^r(T)^{d\times d}}^r + \sum_{F\in\mathcal{F}_T} h_F^{1-r}\|\boldsymbol{v}_F - \boldsymbol{v}_T\|_{L^r(F)^d}^r\right).$$

Discrete operators

For all $T \in \mathcal{T}_h$, we denote $\underline{\boldsymbol{I}}_T^k : W^{1,1}(\Omega)^d \ni \boldsymbol{v} \mapsto (\boldsymbol{\pi}_T^k\boldsymbol{v}, (\boldsymbol{\pi}_F^k\boldsymbol{v})_{F\in\mathcal{F}_T}) \in \underline{\boldsymbol{U}}_T^k$ the interpolation operator and we define the local gradient reconstruction $\mathbf{G}_T^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T)^{d\times d}$ such that

$$\int_T \mathbf{G}_T^k\underline{\boldsymbol{v}}_T : \boldsymbol{\tau} = \int_T \boldsymbol{\nabla}\boldsymbol{v}_T : \boldsymbol{\tau} + \sum_{F\in\mathcal{F}_T}\int_F (\boldsymbol{v}_F - \boldsymbol{v}_T)\cdot(\boldsymbol{\tau}\boldsymbol{n}_{TF}) \qquad \forall \boldsymbol{\tau} \in \mathbb{P}^k(T)^{d\times d}.$$

We also define the local symmetric gradient reconstruction $\mathbf{G}_{\mathrm{s},T}^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T, \mathbb{R}^{d\times d}_{\mathrm{s}})$ and local divergence reconstruction $\mathrm{D}_T^k : \underline{\boldsymbol{U}}_T^k \to \mathbb{P}^k(T)$ such that

$$\mathbf{G}_{\mathrm{s},T}^k = (\mathbf{G}_T^k)_{\mathrm{s}}. \quad \text{and} \quad \mathrm{D}_T^k = \mathrm{tr}(\mathbf{G}_T^k).$$

Finally, we define the global version of these operators such that for all $T \in \mathcal{T}_h$,

$$(\mathbf{G}_h^k\underline{\boldsymbol{v}}_h)_{|T} := \mathbf{G}_T^k\underline{\boldsymbol{v}}_T, \quad (\mathbf{G}_{\mathrm{s},h}^k\underline{\boldsymbol{v}}_h)_{|T} := \mathbf{G}_{\mathrm{s},T}^k\underline{\boldsymbol{v}}_T, \quad (\mathrm{D}_h^k\underline{\boldsymbol{v}}_h)_{|T} := \mathrm{D}_T^k\underline{\boldsymbol{v}}_T, \quad (\underline{\mathbf{I}}_h^k\boldsymbol{v}_h)_{|T} := \underline{\mathbf{I}}_T^k\underline{\boldsymbol{v}}_T.$$

Discrete Problem

The discrete weak formulation reads: Find $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ such that

$$\begin{aligned} \mathrm{a}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) + \mathrm{b}_h(\underline{\boldsymbol{v}}_h, p_h) + \mathrm{c}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{v}}_h) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h & \forall \underline{\boldsymbol{v}}_h \in \underline{\boldsymbol{U}}_{h,0}^k, \\ -\mathrm{b}_h(\underline{\boldsymbol{u}}_h, q_h) &= 0 & \forall q_h \in P_h^k, \end{aligned}$$

where,

- $\mathrm{a}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) := \int_\Omega \boldsymbol{\sigma}(\mathbf{G}_{\mathrm{s},h}^k\underline{\boldsymbol{w}}_h) : \mathbf{G}_{\mathrm{s},h}^k\underline{\boldsymbol{v}}_h + \mathrm{s}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h)$ with s_h a stabilization function,
- $\mathrm{b}_h(\underline{\boldsymbol{v}}_h, q_h) := -\int_\Omega \mathrm{D}_h^k\underline{\boldsymbol{v}}_h\, q_h,$
- $\mathrm{c}_h(\underline{\boldsymbol{w}}_h, \underline{\boldsymbol{v}}_h) := \frac{1}{s}\int_\Omega (\boldsymbol{\chi}(\boldsymbol{w}_h)\cdot\mathbf{G}_h^k)\underline{\boldsymbol{w}}_h\cdot\boldsymbol{v}_h - \frac{1}{s'}\int_\Omega (\boldsymbol{\chi}(\boldsymbol{w}_h)\cdot\mathbf{G}_h^k)\underline{\boldsymbol{v}}_h\cdot\boldsymbol{w}_h + \frac{s-2}{s}\int_\Omega \frac{\boldsymbol{v}_h\cdot\boldsymbol{w}_h}{|\boldsymbol{w}_h|^2}(\boldsymbol{\chi}(\boldsymbol{w}_h)\cdot\mathbf{G}_h^k)\underline{\boldsymbol{w}}_h\cdot\boldsymbol{w}_h.$

Theorem (Well-posedness). There exists a solution to the above discrete problem. Moreover, assuming $2 \leq s \leq \frac{\tilde{r}^*}{\tilde{r}'}$ where $\tilde{r} := \min(2, r)$, and a data smallness condition (cf. [2]), the solution is unique.

Main Results

Theorem (Convergence to minimal regularity solutions). Let $((\underline{\boldsymbol{u}}_h, p_h))_{h\in\mathcal{H}}$ be a sequence of $(\underline{\boldsymbol{U}}_{h,0}^k \times P_h^k)_{h\in\mathcal{H}}$ such that, for all $h \in \mathcal{H}$, $(\underline{\boldsymbol{u}}_h, p_h)$ solves the discrete weak formulation. Assume

$$s < \frac{r^*}{r'}.$$

Then, there exists $(\boldsymbol{u}, p) \in \boldsymbol{W}_0^{1,r}(\Omega)^d \times L_0^{r'}(\Omega)$ solving the weak formulation such that up to a subsequence,

- $\boldsymbol{u}_h \xrightarrow[h\to 0]{} \boldsymbol{u}$ strongly in $L^{[1,r^*)}(\Omega)^d$;
- $p_h \xrightarrow[h\to 0]{} p$ strongly in $L^{r'}(\Omega)$;
- $\mathbf{G}_{\mathrm{s},h}^k\underline{\boldsymbol{u}}_h \xrightarrow[h\to 0]{} \boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}$ strongly in $L^r(\Omega)^{d\times d}$;
- $\mathrm{s}_h(\underline{\boldsymbol{u}}_h, \underline{\boldsymbol{u}}_h) \xrightarrow[h\to 0]{} 0.$

Moreover, if the solution is unique, the convergences extend to the whole sequence.

Theorem (Error estimate). Let $(\boldsymbol{u}, p) \in \boldsymbol{U} \times P$ and $(\underline{\boldsymbol{u}}_h, p_h) \in \underline{\boldsymbol{U}}_{h,0}^k \times P_h^k$ solve the weak and discrete weak formulations, respectively. Assume the uniqueness of such solutions, and $\boldsymbol{u} \in W^{k+2,r}(\mathcal{T}_h)^d \cap W^{k+1,sr'}(\mathcal{T}_h)^d$, $p \in W^{1,r'}(\Omega) \cap W^{k+1,r'}(\mathcal{T}_h)$, $\boldsymbol{\sigma}(\boldsymbol{\nabla}_{\mathrm{s}}\boldsymbol{u}) \in W^{1,r'}(\Omega)^{d\times d} \cap W^{k+1,r'}(\mathcal{T}_h)^{d\times d}$, as well as

$$r \leq 2 \leq s \leq \frac{r^*}{r'},$$

and a data smallness condition depending only on f, $\boldsymbol{\sigma}$, and $\boldsymbol{\chi}$ (cf. [2]). Then,

$$\begin{aligned} \|\underline{\boldsymbol{u}}_h - \underline{\mathbf{I}}_h^k\boldsymbol{u}\|_{\boldsymbol{\varepsilon},r,h} &\leq h^{(k+1)(r-1)}\min(\zeta_h, 1)^{2-r}\mathcal{N}_1 + h^{k+1}\mathcal{N}_2, \\ \|p_h - \pi_h^k p\|_{L^{r'}(\Omega)} &\leq h^{(k+1)(r-1)}\mathcal{N}_3 + h^{(k+1)(r-1)^2}\min(\zeta_h, 1)^{(2-r)(r-1)}\mathcal{N}_4, \end{aligned}$$

where $\zeta_h := h^{k+1}\max_{T\in\mathcal{T}_h}\left(|T|^{-\frac{1}{r}}|\boldsymbol{u}|_{W^{k+2,r}(T)^d}\right)\delta^{-1}$, with $\mathcal{N}_1, \ldots, \mathcal{N}_4 \geq 0$ depending only on u, p, $\boldsymbol{\chi}$, and $\boldsymbol{\sigma}$.

Lid-driven Cavity Application

The domain is the unit square $\Omega = (0,1)^2$, and we enforce a unit tangential velocity $\boldsymbol{u} = (1,0)$ on the top edge and wall boundary conditions on the other edges. We consider $\mu = \frac{2}{\mathrm{Re}}$ with a moderate Reynolds number $\mathrm{Re} = 1000$, $\delta = 1$, and $\nu = 1$. We show the velocity magnitude contours ranging from 0 (blue) to 1 (red). First, we set the convective index $s = 2$ and we vary the viscosity index r:

$r = \frac{3}{2}$ $r = 2$ $r = 3$

Now, we set the viscosity index $r = \frac{5}{2}$ and we vary the convective index s:

$s = \frac{3}{2}$ $s = 2$ $s = \frac{9}{2}$

We observe significant differences in the behaviour of the flow according to the viscous exponent r and the convective exponent s, coherent with the expected physical behaviour. In particular, the viscous effects increase with r, and the turbulent effects increase with s.

References

[1] M. Botti, D. Castanon Quiroz, D. A. Di Pietro, and A. Harnist. "A Hybrid High-Order method for creeping flows of non-Newtonian fluids". In: *ESAIM: Math. Model. Numer. Anal.* 55.5 (2021), pp. 2045–2073. DOI: 10.1051/m2an/2021051.

[2] D. Castanon Quiroz, D. A. Di Pietro, and A. Harnist. "A Hybrid High-Order method for incompressible flows of non-Newtonian fluids with power-like convective behaviour". In: *IMA Journal of Numerical Analysis* (Dec. 2021). Published online. DOI: 10.1093/imanum/drab087.

[3] D. A. Di Pietro and J. Droniou. "$W^{s,p}$-approximation properties of elliptic projectors on polynomial spaces, with application to the error analysis of a Hybrid High-Order discretisation of Leray–Lions problems". In: *Math. Models Methods Appl. Sci.* 27.5 (2017), pp. 879–908. DOI: 10.1142/S0218202517500191.

[4] D. A. Di Pietro and J. Droniou. "A Hybrid High-Order method for Leray–Lions elliptic equations on general meshes". In: *Math. Comp.* 86.307 (2017), pp. 2159–2191. DOI: 10.1090/mcom/3180.

[5] D. A. Di Pietro, J. Droniou, and A. Harnist. "Improved error estimates for Hybrid High-Order discretizations of Leray–Lions problems". In: *Calcolo* 58.19 (2021). DOI: 10.1007/s10092-021-00410-z.

[6] D. A. Di Pietro and S. Krell. "A Hybrid High-Order method for the steady incompressible Navier–Stokes problem". In: *J. Sci. Comput.* 74.3 (2018), pp. 1677–1705. DOI: 10.1007/s10915-017-0512-x.

9th GACM Colloquium on Computational Mechanics, 21-23 September 2022, Essen, Germany