

A Hybrid High-Order method for creeping flows of non-Newtonian fluids

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Abstract

In [1], we develop a Hybrid High-Order method for a modified Stokes problem adapted to non-Newtonian fluids. It is a generalization of the Hybrid High-Order methods implemented to the nonlinear elasticity problem in [2] and to the Navier-Stokes problem in [6] based on the works [3, 4]. The space discretization hinges on local reconstruction operators from hybrid polynomial unknowns at elements and faces. The proposed method has several assets: it is able to handle general polyhedral meshes possibly containing nonmatching interfaces, it allows arbitrary approximation orders, it has a dimension-independent implementation, it allows seamless treatment of nonconforming mesh refinement, it offers stability for inf-sup condition. We give a detailed error estimate in Sobolev-like norms using a discrete Korn's inequality on broken polynomial spaces for the non-Hilbertian case. For the sake of simplicity, we will focus on the power-law fluids since their shear stress-strain rate function verifies the assumptions required to obtain the error estimate.

The Power-Law Stokes Problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, denote a bounded, polyhedral, open star domain with Lipschitz boundary $\partial\Omega$. We consider a power-law fluid occupying Ω characterized by its shear stress-strain rate function $\sigma : \mathbb{R}_s^{d \times d} \rightarrow \mathbb{R}_s^{d \times d}$ which is defined such that for all $\tau \in \mathbb{R}_s^{d \times d}$,

$$\sigma(\tau) = \kappa |\tau|_{d \times d}^{r-2} \tau,$$

where the flow consistency index $\kappa \in (0, +\infty)$ and the flow behavior index $r \in (1, +\infty)$ are two parameters which specify the power-law fluid. Considering that this one is subjected to a volumetric force $f \in L^r(\Omega, \mathbb{R}^d)$, the power-law Stokes problem consists in finding a velocity $u : \Omega \rightarrow \mathbb{R}^d$ and a pressure $p : \Omega \rightarrow \mathbb{R}$ solution of

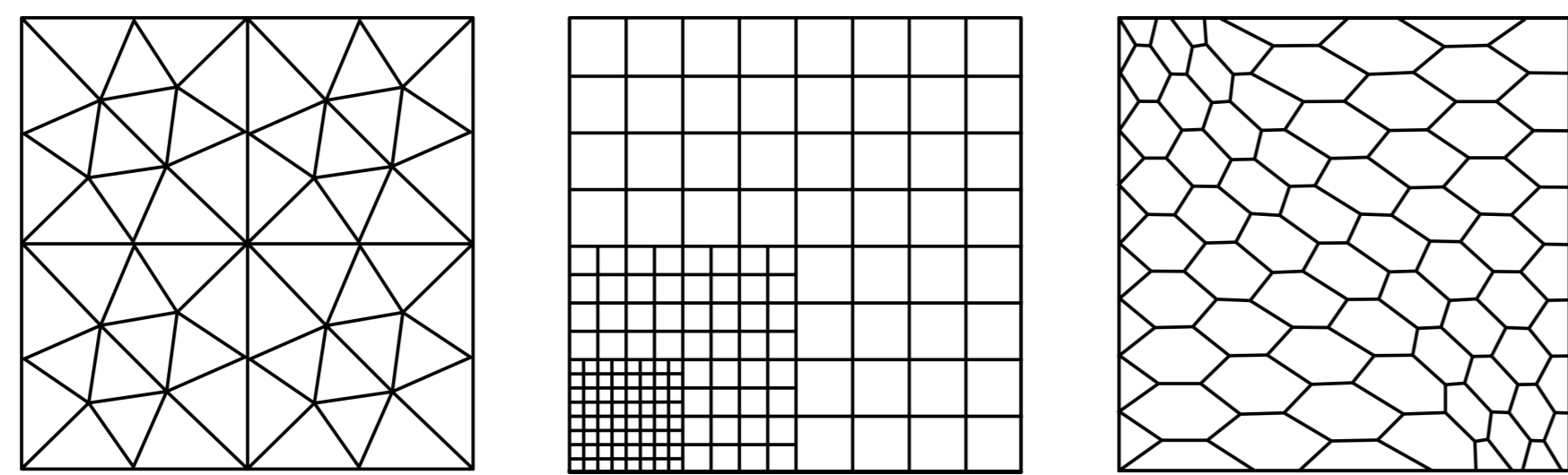
$$\begin{cases} -\nabla \cdot \sigma(\nabla_s u) + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \int_{\Omega} p = 0. \end{cases}$$

It is the modified Stokes problem so that the non-Newtonian character of the fluid is taken into account. In fact, the power-law fluid becomes Newtonian if and only if $r = 2$ and in this case, we obtain the Stokes problem.

Hybrid High-Order

Features

- Able to handle general polyhedral meshes possibly containing nonmatching interfaces



- Dimension-independent implementation
- Arbitrary order: better accuracy for a fixed mesh or fewer elements for a given precision
- Seamless treatment of nonconforming mesh refinement
- Offers stability for inf-sup condition

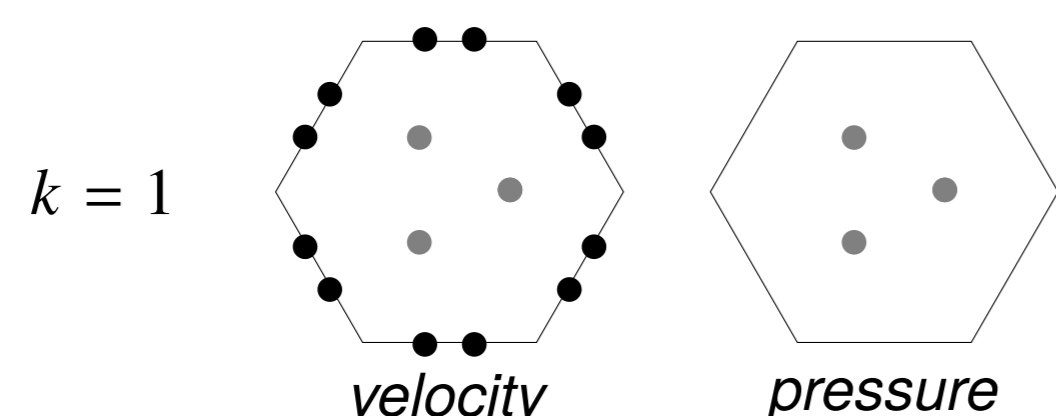
Hybrid spaces

The discrete local and global spaces containing velocity unknowns are respectively defined as

$$\underline{U}_T^k := \mathbb{P}^k(T, \mathbb{R}^d) \times \left(\prod_{F \in \mathcal{F}_T} \mathbb{P}^k(F, \mathbb{R}^d) \right), \quad \underline{U}_h^k := \prod_{T \in \mathcal{T}_h} \underline{U}_T^k.$$

For all $\underline{v}_h \in \underline{U}_h^k$, we use the discrete notation $\underline{v}_h := (\underline{v}_T)_{T \in \mathcal{T}_h}$ where $\underline{v}_T := (v_T, (v_F)_{F \in \mathcal{F}_T}) \in \underline{U}_T^k$ for all element T of the mesh \mathcal{T}_h . The discrete space containing pressure unknowns is defined by

$$P_h^k := \mathbb{P}^k(\mathcal{T}_h, \mathbb{R}) \cap L_0^r(\Omega, \mathbb{R}).$$



We define on the zero boundary subspace $\underline{U}_{h,0}^k$ the norm $\|\cdot\|_{\varepsilon,r,h}$ such that for all $\underline{v}_h \in \underline{U}_{h,0}^k$,

$$\|\underline{v}_h\|_{\varepsilon,r,h} := \left(\sum_{T \in \mathcal{T}_h} \left(\|\nabla_s v_T\|_{L^r(T, \mathbb{R}_s^{d \times d})}^r + \sum_{F \in \mathcal{F}_T} h_F^{1-r} \|v_F - v_T\|_{L^r(F, \mathbb{R}^d)}^r \right) \right)^{\frac{1}{r}}.$$

We have a **discrete Korn's inequality**: there exists $C_K > 0$ independent of h such that, for all $\underline{v}_h \in \underline{U}_{h,0}^k$,

$$\|\underline{v}_h\|_{L^r(\Omega, \mathbb{R}^d)} + \|v_h\|_{W^{1,r}(\mathcal{T}_h, \mathbb{R}^d)} \leq C_K \|\underline{v}_h\|_{\varepsilon,r,h}.$$

Discrete operators

We denote $\underline{I}_h^k : W^{1,1}(\Omega, \mathbb{R}^d) \ni v \mapsto (\pi_T^k v, (\pi_F^k v)_{F \in \mathcal{F}_T})_{T \in \mathcal{T}_h} \in \underline{U}_h^k$ the interpolation operator. For all element $T \in \mathcal{T}_h$, we define

- the local symmetric gradient operator $G_{s,T}^k : \underline{U}_T^k \rightarrow \mathbb{P}^k(T, \mathbb{R}_s^{d \times d})$ such that, for all $\underline{v}_T \in \underline{U}_T^k$,
- $$\int_T G_{s,T}^k \underline{v}_T : \tau = \int_T \nabla_s v_T : \tau + \sum_{F \in \mathcal{F}_T} \int_F (v_F - v_T) \cdot (\tau n_{TF}) \quad \forall \tau \in \mathbb{P}^k(T, \mathbb{R}_s^{d \times d}),$$

- the local divergence operator $D_T^k : \underline{U}_T^k \rightarrow \mathbb{P}^k(T, \mathbb{R})$ by $D_T^k = \text{tr}(G_{s,T}^k)$,
- the local symmetric reconstruction operator $r_{s,T}^{k+1} : \underline{U}_T^k \rightarrow \mathbb{P}^{k+1}(T, \mathbb{R}^d)$ such that for all $\underline{v}_T \in \underline{U}_T^k$,

$$\begin{cases} \int_T (\nabla_s r_{s,T}^{k+1} \underline{v}_T - G_{s,T}^k \underline{v}_T) : \nabla_s w = 0 \quad \forall w \in \mathbb{P}^{k+1}(T, \mathbb{R}^d), \\ \int_T r_{s,T}^{k+1} \underline{v}_T = \int_T v_T, \quad \int_T \nabla_{ss} r_{s,T}^{k+1} \underline{v}_T = \frac{1}{2} \sum_{F \in \mathcal{F}_T} \int_F (n_{TF} \otimes v_F - v_F \otimes n_{TF}). \end{cases}$$

Discrete problem

We define the stabilization function $s_T : \underline{U}_T^k \times \underline{U}_T^k \rightarrow \mathbb{R}$ such that for all $\underline{v}_T, \underline{w}_T \in \underline{U}_T^k$,

$$s_T(\underline{w}_T, \underline{v}_T) := \sum_{F \in \mathcal{F}_T} h_F^{1-r} \int_F |\Delta_{TF}^k \underline{w}_T|^{r-2} \Delta_{TF}^k \underline{w}_T \cdot \Delta_{TF}^k \underline{v}_T$$

where $\Delta_{TF}^k \underline{v}_T := \pi_F^k(r_{s,T}^{k+1} \underline{v}_T - v_T) - \pi_T^k(r_{s,T}^{k+1} \underline{v}_T - v_T)$ for all $\underline{v}_T \in \underline{U}_T^k$ and all $F \in \mathcal{F}_T$. We define the function $a_h : \underline{U}_h^k \times \underline{U}_h^k \rightarrow \mathbb{R}$ such that for all $\underline{w}_h, \underline{v}_h \in \underline{U}_h^k$,

$$a_h(\underline{w}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} \left(\int_T \sigma(G_{s,T}^k \underline{w}_T) : G_{s,T}^k \underline{v}_T + \kappa(r-1) s_T(\underline{w}_T, \underline{v}_T) \right).$$

The function a_h has **coercivity**, **boundedness** and **consistency** properties. Indeed, for all $\underline{v}_h, \underline{w}_h \in \underline{U}_{h,0}^k$ and all $w \in W_0^{1,r}(\Omega, \mathbb{R}^d) \cap W^{k+2,r}(\mathcal{T}_h, \mathbb{R}^d)$, there exist $C_{cv}, C_{gr}, C_{a1}(w), C_{a2}(w) > 0$ independent of h such that,

$$\begin{aligned} a_h(\underline{v}_h, \underline{v}_h) &\geq C_{cv} \|\underline{v}_h\|_{\varepsilon,r,h}^r, \quad a_h(\underline{w}_h, \underline{v}_h) \leq C_{gr} \|\underline{w}_h\|_{\varepsilon,r,h}^{r-1} \|\underline{v}_h\|_{\varepsilon,r,h}, \\ \sup_{\underline{v}_h \in \underline{U}_{h,0}^k, \|\underline{v}_h\|_{\varepsilon,r,h}=1} \left| \int_{\Omega} v_h \cdot (\nabla \cdot \sigma(\nabla_s w)) + a_h(\underline{I}_h^k w, \underline{v}_h) \right| &\leq C_{a1}(w) h^{k+1} + C_{a2}(w) h^{(k+1)(r-1)}. \end{aligned}$$

We define the bilinear form $b_h : \underline{U}_h^k \times \mathbb{P}^k(\mathcal{T}_h, \mathbb{R}) \rightarrow \mathbb{R}$ such that, for all $(\underline{v}_h, q_h) \in \underline{U}_h^k \times \mathbb{P}^k(\mathcal{T}_h, \mathbb{R})$,

$$b_h(\underline{v}_h, q_h) := - \sum_{T \in \mathcal{T}_h} \int_T D_T^k \underline{v}_T (q_h)|_T.$$

We can prove an **inf-sup condition** and a **consistency** property for the function b_h . Indeed, for all $q_h \in P_h^k$ and all $q \in W^{1,r'}(\Omega, \mathbb{R}) \cap W^{k+1,r'}(\mathcal{T}_h, \mathbb{R})$, there exists $C_b(q) > 0$ independent of h such that,

$$\|q_h\|_{L^r(\Omega, \mathbb{R})} \leq \sup_{\underline{v}_h \in \underline{U}_{h,0}^k, \|\underline{v}_h\|_{\varepsilon,r,h}=1} b_h(\underline{v}_h, q_h), \quad \sup_{\underline{v}_h \in \underline{U}_{h,0}^k, \|\underline{v}_h\|_{\varepsilon,r,h}=1} \left| \int_{\Omega} \nabla_h q \cdot v_h - b_h(\underline{v}_h, \pi_h^k q) \right| \leq C_b(q) h^{k+1}.$$

The discrete problem reads: Find $(\underline{u}_h, p_h) \in \underline{U}_{h,0}^k \times P_h^k$ such that,

$$\begin{cases} a_h(\underline{u}_h, \underline{v}_h) + b_h(\underline{v}_h, p_h) = \int_{\Omega} f \cdot v_h & \forall \underline{v}_h \in \underline{U}_{h,0}^k, \\ -b_h(\underline{u}_h, q_h) = 0 & \forall q_h \in P_h^k. \end{cases}$$

Error estimate

Theorem: Let $(u, p) \in W_0^{1,r}(\Omega, \mathbb{R}^d) \times L^r(\Omega, \mathbb{R})$ and $(\underline{u}_h, p_h) \in \underline{U}_{h,0}^k \times P_h^k$ the solutions of the continuous and discrete problem. Assume the regularity $(u, p) \in W^{k+2,r}(\mathcal{T}_h, \mathbb{R}^d) \times (W^{1,r'}(\Omega, \mathbb{R}) \cap W^{k+1,r'}(\mathcal{T}_h, \mathbb{R}))$. Then, setting $\underline{e}_h := \underline{u}_h - \underline{I}_h^k u$ and $e_h := p_h - \pi_h^k p$, there exists $C_1(u, p), \dots, C_5(u, p) > 0$ independent of h such that

$$\begin{aligned} \|\underline{e}_h\|_{\varepsilon,r,h} &\leq C_1(u, p) h^{k+1} + C_2(u, p) h^{(k+1) \min(\frac{1}{r-1}, r-1)}, \\ \|\underline{e}_h\|_{L^r(\Omega, \mathbb{R}^d)} &\leq C_3(u, p) h^{k+1} + C_4(u, p) h^{(k+1) \min(\frac{1}{r-1}, r-1)} + C_5(u, p) h^{(k+1)(r-1) \max(1, r-1)}. \end{aligned}$$

We can deduce that, the discrete velocity converges with an asymptotic order of

$$O_{\text{vel}}^k := \begin{cases} \frac{k+1}{r-1} & \text{if } r \geq 2, \\ (k+1)(r-1) & \text{if } r < 2, \end{cases}$$

and the pressure converges with an asymptotic order of

$$O_{\text{pre}}^k := \begin{cases} \frac{k+1}{r-1} & \text{if } r \geq 2, \\ (k+1)(r-1)^2 & \text{if } r < 2. \end{cases}$$

Numerical Results

We demonstrate the asymptotic order of convergence for the velocity given by the above theorem. We solve the non-linear elasticity problem on the unit square $\Omega = (0, 1)^2$ with, for all $(x, y) \in \Omega$,

$$u(x, y) = (\cos(x)e^y, \sin(x)e^y).$$

The parameter κ is taken equal to 1 and we vary the flow behavior index $r \in (1, +\infty)$.

k = 1											
ndofs	nnz	$r = \frac{5}{4}$ $O_{\text{vel}}^k = \frac{1}{2}$		$r = 2$ $O_{\text{vel}}^k = 2$		$r = 3$ $O_{\text{vel}}^k = 1$					
		$\ \underline{e}_h\ _{\varepsilon,r,h}$	EOC	$\ \underline{e}_h\ _{\varepsilon,r,h}$	EOC	$\ \underline{e}_h\ _{\varepsilon,r,h}$	EOC				
268	4848	5.39e-05	-	268	4848	1.59e-03	-	268	4848	4.63e-02	-
1148	21936	6.36e-06	3.08	1148	21936	4.31e-04	1.89	1148	21936	1.79e-02	1.38
4732	92592	6.73e-07	3.24	4732	92592	1.08e-04	2.00	4732	92592	7.24e-03	1.30
18548	366864	7.80e-08	3.11	18548	366864	2.83e-05	1.93	18548	366864	2.96e-03	1.29
76096	1513728	7.89e-09	3.30	76096	1513728	7.00e-06	2.02	76096	1513728	1.17e-03	1.34

k = 2											
ndofs	nnz	$r = \frac{5}{4}$ $O_{\text{vel}}^k = \frac{3}{4}$		$r = 2$ $O_{\text{vel}}^k = 3$		$r = 3$ $O_{\text{vel}}^k = \frac{3}{2}$					
		$\ \underline{e}_h\ _{\varepsilon,r,h}$	EOC	$\ \underline{e}_h\ _{\varepsilon,r,h}$	EOC	$\ \underline{e}_h\ _{\varepsilon,r,h}$	EOC				
268	4848	1.24e-07	-	268	4848	4.14e-05	-	268	4848	6.73e-03	-
1148	21936	5.02e-09	4.63	1148	21936	5.47e-06	2.92	1148	21936	1.56e-03	2.11
4732	92592	1.52e-10	5.05	4732	92592	6.83e-07	3.00	4732	92592	4.16e-04	1.91
18548	366864	6.34e-12	4.58	18548	366864	9.12e-08	2.90	18548	366864	1.07e-04	1.95
76096	1513728	2.15e-13	4.88	76096	1513728	1.12e-08	3.03	76096	1513728	2.68e-05	2.01

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