

# A HYBRID HIGH-ORDER METHOD FOR CREEPING FLOWS OF NON-NEWTONIAN FLUIDS

<sup>1</sup>André Harnist

with

<sup>2</sup>Michele Botti, <sup>1</sup>Daniel Castanon Quiroz, <sup>1</sup>Daniele A. Di Pietro

<sup>1</sup>IMAG, University of Montpellier, CNRS, Montpellier, France

<sup>2</sup>MOX, Department of Mathematics, Politecnico di Milano, Milano, Italy

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# Outline

1. The Hybrid High-Order method
2. Newtonian and non-Newtonian fluids
3. The Stokes equations
4. Discretization with the HHO method

## Hybrid High-Order (HHO)

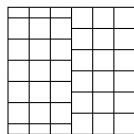
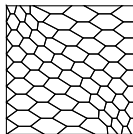
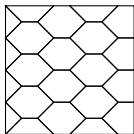
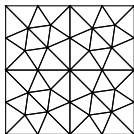
**Hybrid:** two kinds of unknowns located on the mesh and its skeleton.

**High-Order:** the unknowns live in broken polynomial spaces of degree  $k \in \mathbb{N}$ .

# General features

This approach possesses several attractive features:

- ▶ Arbitrary **approximation order** ( $k \geq 0$ ).
- ▶ Formulation valid for arbitrary **space dimension**.
- ▶ Seamless treatment of **nonconforming** mesh refinement.



- ▶ Moderate computational costs thanks to **static condensation**.
- ▶ **Inf-sup stable** discretizations.

# References

## Book.

D. A. Di Pietro and J. Droniou.

**The Hybrid High-Order method for polytopal meshes.**

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More information: [andrearnist.fr](http://andrearnist.fr)

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# Newtonian and non-Newtonian fluids

We can distinguish fluids according to their viscosity:

- ▶ **Newtonian**: viscosity is **constant** with increased stress (e.g. air, water).
- ▶ **non-Newtonian**:
  - ▶ **Pseudoplastic (shear thinning)**: viscosity **decreases** with increased stress (e.g. blood, honey).
  - ▶ **Dilatant (shear thickening)**: viscosity **increases** with increased stress (e.g. oobleck, quicksand).



# Newtonian fluid

We characterize the movement of a fluid with a **strain-stress function**

$$\sigma : \Omega \times \mathbb{R}_s^{d \times d} \rightarrow \mathbb{R}_s^{d \times d}$$

where  $\mathbb{R}_s^{d \times d} := \{\boldsymbol{\tau} \in \mathbb{R}^{d \times d} : \boldsymbol{\tau}^T = \boldsymbol{\tau}\}$ ,  $d \in \{2, 3\}$ .

- ▶ A **Newtonian** fluid is one for which the law  $\sigma$  is linear.
- ▶ For the **non-Newtonian** fluids, several laws model them:
  - ▶ Power-law
  - ▶ Carreau–Yasuda
  - ▶ Yeleswarapu
  - ▶ Quemada
  - ▶ Cross
  - ▶ ...

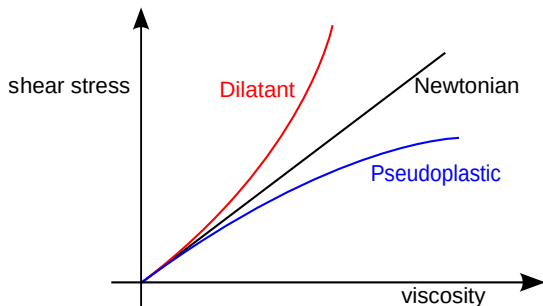
# Power-law

A **power-law** fluid is one for which  $\sigma$  is such that,

$$\sigma(\boldsymbol{\tau}) = \mu |\boldsymbol{\tau}|_{d \times d}^{r-2} \boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in \mathbb{R}_s^{d \times d},$$

where  $\mu > 0$  is the flow consistency index and  $r > 1$  is the flow behavior index.

- ▶ If  $r < 2$ , the fluid is **pseudoplastic (shear thinning)**.
- ▶ If  $r = 2$ , the fluid is **Newtonian**.
- ▶ If  $r > 2$ , the fluid is **dilatant (shear thickening)**.



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# The steady generalized Stokes problem

Let  $\Omega \subset \mathbb{R}^d$  denote a bounded, connected polyhedral open set with Lipschitz boundary  $\partial\Omega$ .

The steady **generalized Stokes** problem reads: Find  $\mathbf{u}$  and  $p$  such that

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma}(\cdot, \nabla_s \mathbf{u}) + \nabla p &= \mathbf{f} && \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, \\ \int_{\Omega} p &= 0, \end{aligned}$$

where,

- ▶  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^d$  is the **force** applied on the fluid,
- ▶  $\mathbf{u} : \Omega \rightarrow \mathbb{R}^d$  is the **velocity** of the fluid and  $\nabla_s \mathbf{u} := \frac{\nabla \mathbf{u} + \nabla^T \mathbf{u}}{2}$ ,
- ▶  $p : \Omega \rightarrow \mathbb{R}$  is the **pressure** of the fluid,
- ▶  $\boldsymbol{\sigma} : \Omega \times \mathbb{R}_s^{d \times d} \rightarrow \mathbb{R}_s^{d \times d}$  is the strain-stress law of the fluid.

# Assumptions on $\sigma$ - power-framed function

## Assumption. (power-framed)

There exists  $r \in (1, +\infty)$  such that:

- ▶  $\sigma : \Omega \times \mathbb{R}_s^{d \times d} \rightarrow \mathbb{R}_s^{d \times d}$  is measurable,
- ▶  $\sigma(\cdot, \mathbf{0}) \in L^{r'}(\Omega, \mathbb{R}_s^{d \times d})$  a.e. in  $\Omega$  where  $r' := \frac{r}{r-1}$ ,
- ▶  $\sigma$  is  $r$ -power-framed: defining the singular exponent of  $r$  by

$$\tilde{r} := \min(r, 2),$$

there is  $\sigma_{\text{de}}, \sigma_{\text{hc}}, \sigma_{\text{sm}} \in \mathbb{R}^+ \times \mathbb{R}_*^+ \times \mathbb{R}_*^+$  s.t.

$$|\sigma(\mathbf{x}, \boldsymbol{\tau}) - \sigma(\mathbf{x}, \boldsymbol{\eta})|_{d \times d} \leq \sigma_{\text{hc}} (\sigma_{\text{de}}^r + |\boldsymbol{\tau}|_{d \times d}^r + |\boldsymbol{\eta}|_{d \times d}^r)^{\frac{r-\tilde{r}}{r}} |\boldsymbol{\tau} - \boldsymbol{\eta}|_{d \times d}^{\tilde{r}-1},$$

$$(\sigma(\mathbf{x}, \boldsymbol{\tau}) - \sigma(\mathbf{x}, \boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \geq \sigma_{\text{sm}} (\sigma_{\text{de}}^r + |\boldsymbol{\tau}|_{d \times d}^r + |\boldsymbol{\eta}|_{d \times d}^r)^{\frac{\tilde{r}-2}{r}} |\boldsymbol{\tau} - \boldsymbol{\eta}|_{d \times d}^{r+2-\tilde{r}},$$

for all  $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}_s^{d \times d}$  and a.e.  $\mathbf{x} \in \Omega$ .

# Weak formulation

Assuming  $\mathbf{f} \in L^r(\Omega, \mathbb{R}^d)$ , we define

- ▶  $U := W_0^{1,r}(\Omega, \mathbb{R}^d) = \{\mathbf{v} \in W^{1,r}(\Omega, \mathbb{R}^d) : \mathbf{v}|_{\partial\Omega} = \mathbf{0}\},$
- ▶  $P := L_0^r(\Omega, \mathbb{R}) = \{q \in L^r(\Omega, \mathbb{R}) : \int_{\Omega} q = 0\}.$

The **weak formulation** of the Stokes problem reads:  
Find  $(\mathbf{u}, p) \in U \times P$  such that

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v} & \forall \mathbf{v} \in U, \\ -b(\mathbf{u}, q) &= 0 & \forall q \in P, \end{aligned}$$

where, for all  $\mathbf{v}, \mathbf{w} \in U$  and all  $q \in P$ ,

$$\begin{aligned} a(\mathbf{w}, \mathbf{v}) &:= \int_{\Omega} \boldsymbol{\sigma}(\cdot, \nabla_s \mathbf{w}) : \nabla_s \mathbf{v}, \\ b(\mathbf{v}, q) &:= - \int_{\Omega} (\nabla \cdot \mathbf{v}) q. \end{aligned}$$

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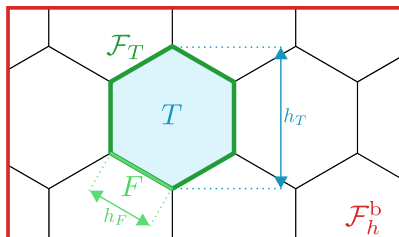
# Mesh and notations

Let  $h \in (0, +\infty)$ . We define a mesh of  $\Omega$  as a couple  $(\mathcal{T}_h, \mathcal{F}_h)$  such that

- ▶  $\mathcal{T}_h$  is a finite collection of polyhedral elements  $T$  with diameter  $h_T$ ,
- ▶  $\mathcal{F}_h$  is a finite collection of planar faces  $F$  with diameter  $h_F$ ,
- ▶  $\bigcup_{T \in \mathcal{T}_h} \bar{T} = \bar{\Omega}$  and  $\max_{T \in \mathcal{T}_h} h_T = h$ ,
- ▶  $(\mathcal{T}_h, \mathcal{F}_h)$  satisfies some geometrical requirements...

We also define the following subsets of  $\mathcal{F}_h$ :

- ▶  $\mathcal{F}_h^b := \{F \in \mathcal{F}_h : F \subset \partial\Omega\}$ ,
- ▶  $\mathcal{F}_T := \{F \in \mathcal{F}_h : F \subset \partial T\}$  for all  $T \in \mathcal{T}_h$ .





# Discrete spaces and notations

Let  $k \geq 1$  be the polynomial degree of the HHO method.

- ▶ For all  $T \in \mathcal{T}_h$ , we define the discrete **local** space:

$$\underline{U}_T^k := \mathbb{P}^k(T, \mathbb{R}^d) \times \left( \prod_{F \in \mathcal{F}_T} \mathbb{P}^k(F, \mathbb{R}^d) \right).$$

We use the discrete notation  $\underline{\mathbf{v}}_T := (\mathbf{v}_T, (\mathbf{v}_F)_{F \in \mathcal{F}_T}) \in \underline{U}_T^k$ .

- ▶ We define the discrete **global** space:

$$\underline{U}_h^k := \prod_{T \in \mathcal{T}_h} \underline{U}_T^k.$$

We use the discrete notation  $\underline{\mathbf{v}}_h := (\underline{\mathbf{v}}_T)_{T \in \mathcal{T}_h} \in \underline{U}_h^k$ .

- ▶ We define the **interpolation operator**  $\underline{\mathbf{I}}_h^k : W^{1,1}(\Omega, \mathbb{R}^d) \rightarrow \underline{U}_h^k$  s.t.

$$\underline{\mathbf{I}}_h^k \mathbf{v} := (\pi_T^k \mathbf{v}|_T, (\pi_F^k \mathbf{v}|_F)_{F \in \mathcal{F}_T})_{T \in \mathcal{T}_h}.$$

# Spaces and norms of discrete unknowns

- ▶ The discrete space containing **velocity** unknowns is defined by

$$\underline{U}_{h,0}^k := \{ \underline{\mathbf{v}}_h \in \underline{U}_h^k : \mathbf{v}_F = \mathbf{0} \quad \forall F \in \mathcal{F}_h^b \}.$$

We endow  $\underline{U}_{h,0}^k$  with the semi-norm  $\|\cdot\|_{\varepsilon,r,h}$  defined by

$$\|\underline{\mathbf{v}}_h\|_{\varepsilon,r,h}^r := \sum_{T \in \mathcal{T}_h} \left( \|\nabla_s \mathbf{v}_T\|_{L^r(T, \mathbb{R}^{d \times d})}^r + \sum_{F \in \mathcal{F}_T} h_F^{1-r} \|\mathbf{v}_F - \mathbf{v}_T\|_{L^r(F, \mathbb{R}^d)}^r \right).$$

- ▶ The discrete space containing **pressure** unknowns is defined by

$$P_h^k := \left\{ q_h \in L_0^{r'}(\Omega, \mathbb{R}) : (q_h)|_T \in \mathbb{P}^k(T, \mathbb{R}) \quad \forall T \in \mathcal{T}_h \right\}.$$

We endow  $P_h^k$  with the norm  $\|\cdot\|_{L^{r'}(\Omega, \mathbb{R})}$ .

# Korn and discrete Korn inequalities

The regularity of  $\Omega$  yields the following **Korn inequality**:

$$\|\mathbf{v}\|_{W^{1,r}(\Omega, \mathbb{R}^d)} \lesssim \|\nabla_s \mathbf{v}\|_{L^r(\Omega, \mathbb{R}^{d \times d})} \quad \forall \mathbf{v} \in W_0^{1,r}(\Omega, \mathbb{R}^d).$$

## **Theorem.** *(discrete Korn inequality)*

It holds, with hidden constant depending only on  $\Omega, d, k, \rho$  and  $r$ ,

$$\|\mathbf{v}_h\|_{W^{1,r}(\mathcal{T}_h, \mathbb{R}^d)} \lesssim \|\underline{\mathbf{v}}_h\|_{\varepsilon, r, h} \quad \forall \underline{\mathbf{v}}_h \in \underline{U}_{h,0}^k,$$

As a consequence,  $\|\cdot\|_{\varepsilon, r, h}$  is a **norm** on  $\underline{U}_{h,0}^k$ .

Hilbertian case  $r = 2 \rightsquigarrow$  see [Botti, Di Pietro, Guglielmana; 2019]

# Discrete operators

For all  $T \in \mathcal{T}_h$ , we define:

- ▶ the **discrete local symmetric gradient**  $\mathbf{G}_{s,T}^k : \underline{\mathbf{U}}_T^k \longrightarrow \mathbb{P}^k(T, \mathbb{R}_s^{d \times d})$  such that, for all  $\underline{\mathbf{v}}_T \in \underline{\mathbf{U}}_T^k$ ,

$$\int_T \mathbf{G}_{s,T}^k \underline{\mathbf{v}}_T : \boldsymbol{\tau} = \int_T \nabla_s \mathbf{v}_T : \boldsymbol{\tau} + \sum_{F \in \mathcal{F}_T} \int_F (\mathbf{v}_F - \mathbf{v}_T) \cdot (\boldsymbol{\tau} \mathbf{n}_{TF}) \quad \forall \boldsymbol{\tau} \in \mathbb{P}^k(T, \mathbb{R}_s^{d \times d}).$$

- ▶ the **discrete local divergence**  $\mathbf{D}_T^k : \underline{\mathbf{U}}_T^k \longrightarrow \mathbb{P}^k(T, \mathbb{R})$  as the trace of the discrete gradient operator:  $\mathbf{D}_T^k = \text{tr}(\mathbf{G}_{s,T}^k)$ .

The global versions of these operators are defined by: for all  $\underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_h^k$ ,

$$(\mathbf{G}_{s,h}^k \underline{\mathbf{v}}_h)|_T := \mathbf{G}_{s,T}^k \underline{\mathbf{v}}_T \quad \forall T \in \mathcal{T}_h,$$

$$(\mathbf{D}_h^k \underline{\mathbf{v}}_h)|_T := \mathbf{D}_T^k \underline{\mathbf{v}}_T \quad \forall T \in \mathcal{T}_h.$$

# Discrete weak formulation

The **discrete weak formulation** reads: Find  $(\underline{\mathbf{u}}_h, p_h) \in \underline{\mathbf{U}}_{h,0}^k \times P_h^k$  s.t.

$$\begin{aligned} \mathbf{a}_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) + \mathbf{b}_h(\underline{\mathbf{v}}_h, p_h) &= \int_{\Omega} \mathbf{f} \cdot \underline{\mathbf{v}}_h & \forall \underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_{h,0}^k, \\ -\mathbf{b}_h(\underline{\mathbf{u}}_h, q_h) &= 0 & \forall q_h \in P_h^k, \end{aligned}$$

where, for all  $\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h \in \underline{\mathbf{U}}_h^k$  and  $q_h \in P_h^k$ ,

- ▶  $\mathbf{a}_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h) := \int_{\Omega} \boldsymbol{\sigma}(\cdot, \mathbf{G}_{s,h}^k \underline{\mathbf{w}}_h) : \mathbf{G}_{s,h}^k \underline{\mathbf{v}}_h + s_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h),$
- ▶  $\mathbf{b}_h(\underline{\mathbf{v}}_h, q_h) := - \int_{\Omega} \mathbf{D}_h^k \underline{\mathbf{v}}_h q_h.$

and where  $s_h$  is a classic HHO stabilization function satisfying a power-framed assumption similar to that of  $\boldsymbol{\sigma}$ .

# Assumptions on $s_h$

## Assumption. (Stabilization function $s_h$ )

- ▶ For all  $\mathbf{v}_h \in \underline{U}_h^k$ ,  $s_h(\mathbf{v}_h, \cdot)$  is **linear**.
- ▶ **Polynomial consistency**. For all  $\mathbf{v}_h \in \underline{U}_h^k$  and  $\mathbf{w} \in \mathbb{P}^{k+1}(T, \mathbb{R}^d)$ ,

$$s_T(\underline{\mathbf{I}}_T \mathbf{w}, \mathbf{v}_T) = 0.$$

- ▶ **Stability and boundedness**. For all  $\mathbf{v}_h \in \underline{U}_h^k$ ,

$$\|\mathbf{G}_{s,h}^k \mathbf{v}_h\|_{L^r(\Omega, \mathbb{R}^{d \times d})}^r + s_h(\mathbf{v}_h, \mathbf{v}_h) \simeq \|\mathbf{v}_h\|_{\varepsilon, r, h}^r.$$

- ▶  **$r$ -power-framed**. For all  $\underline{\mathbf{u}}_h, \mathbf{v}_h, \underline{\mathbf{w}}_h \in \underline{U}_h^k$  with  $\underline{\mathbf{e}}_h := \underline{\mathbf{u}}_h - \underline{\mathbf{w}}_h$ ,

$$|s_h(\underline{\mathbf{u}}_h, \mathbf{v}_h) - s_h(\underline{\mathbf{w}}_h, \mathbf{v}_h)| \lesssim (s_h(\underline{\mathbf{u}}_h, \underline{\mathbf{u}}_h) + s_h(\underline{\mathbf{w}}_h, \underline{\mathbf{w}}_h))^{\frac{r-\bar{r}}{r}} s_h(\underline{\mathbf{e}}_h, \underline{\mathbf{e}}_h)^{\frac{\bar{r}-1}{r}} s_h(\mathbf{v}_h, \mathbf{v}_h)^{\frac{1}{r}},$$
$$s_h(\underline{\mathbf{u}}_h, \underline{\mathbf{e}}_h) - s_h(\underline{\mathbf{w}}_h, \underline{\mathbf{e}}_h) \gtrsim (s_h(\underline{\mathbf{u}}_h, \underline{\mathbf{u}}_h) + s_h(\underline{\mathbf{w}}_h, \underline{\mathbf{w}}_h))^{\frac{\bar{r}-2}{r}} s_h(\underline{\mathbf{e}}_h, \underline{\mathbf{e}}_h)^{\frac{r+2-\bar{r}}{r}}.$$

# Properties of $a_h$

## Lemme. (Properties of $a_h$ )

- ▶ Hölder continuity. For all  $\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h, \underline{\mathbf{w}}_h \in \underline{\mathbf{U}}_h^k$ , setting  $\underline{\mathbf{e}}_h := \underline{\mathbf{u}}_h - \underline{\mathbf{w}}_h$ ,

$$|a_h(\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h) - a_h(\underline{\mathbf{w}}_h, \underline{\mathbf{v}}_h)| \lesssim \sigma_{\text{hc}} \left( \sigma_{\text{de}}^r + \|\underline{\mathbf{u}}_h\|_{\varepsilon, r, h}^r + \|\underline{\mathbf{w}}_h\|_{\varepsilon, r, h}^r \right)^{\frac{r-\tilde{r}}{r}} \|\underline{\mathbf{e}}_h\|_{\varepsilon, r, h}^{\tilde{r}-1} \|\underline{\mathbf{v}}_h\|_{\varepsilon, r, h}.$$

- ▶ Strong monotonicity. For all  $\underline{\mathbf{u}}_h, \underline{\mathbf{v}}_h, \underline{\mathbf{w}}_h \in \underline{\mathbf{U}}_h^k$ , setting  $\underline{\mathbf{e}}_h := \underline{\mathbf{u}}_h - \underline{\mathbf{w}}_h$ ,

$$a_h(\underline{\mathbf{u}}_h, \underline{\mathbf{e}}_h) - a_h(\underline{\mathbf{w}}_h, \underline{\mathbf{e}}_h) \gtrsim \sigma_{\text{sm}} \left( \sigma_{\text{de}}^r + \|\underline{\mathbf{u}}_h\|_{\varepsilon, r, h}^r + \|\underline{\mathbf{w}}_h\|_{\varepsilon, r, h}^r \right)^{\frac{2-\tilde{r}}{r}} \|\underline{\mathbf{e}}_h\|_{\varepsilon, r, h}^{r+2-\tilde{r}}.$$

# Properties of $\mathbf{b}_h$

## Lemme. (Properties of $\mathbf{b}_h$ )

- ▶ **Inf-sup stability.** For all  $q_h \in P_h^k$ ,

$$\|q_h\|_{L^r(\Omega, \mathbb{R})} \lesssim \sup_{\mathbf{v}_h \in \underline{\mathbf{U}}_{h,0}^k, \|\mathbf{v}_h\|_{\varepsilon,r,h}=1} \mathbf{b}_h(\mathbf{v}_h, q_h).$$

- ▶ **Fortin operator.** For all  $\mathbf{v} \in W^{1,r}(\Omega, \mathbb{R}^d)$ ,

$$\begin{aligned} \|\mathbf{I}_h^k \mathbf{v}\|_{\varepsilon,r,h} &\lesssim |\mathbf{v}|_{W^{1,r}(\Omega, \mathbb{R}^d)}, \\ \mathbf{b}_h(\mathbf{I}_h^k \mathbf{v}, q_h) &= b(\mathbf{v}, q_h) \quad \forall q_h \in \mathbb{P}^k(\mathcal{T}_h, \mathbb{R}). \end{aligned}$$



# Well-posedness and a priori bounds

## Theorem. (Well-posedness and a priori bounds)

There exists a unique solution  $(\underline{\mathbf{u}}_h, p_h) \in \underline{\mathbf{U}}_{h,0}^k \times P_h^k$  to the discrete weak formulation. Additionally, the following a priori bounds hold:

$$\|\underline{\mathbf{u}}_h\|_{\varepsilon,r,h} \lesssim \left( \sigma_{\text{sm}}^{-1} \|\mathbf{f}\|_{L'(\Omega, \mathbb{R}^d)} \right)^{\frac{1}{r-1}} + \left( \sigma_{\text{de}}^{2-\tilde{r}} \sigma_{\text{sm}}^{-1} \|\mathbf{f}\|_{L'(\Omega, \mathbb{R}^d)} \right)^{\frac{1}{r+1-\tilde{r}}},$$
$$\|p_h\|_{L'(\Omega, \mathbb{R})} \lesssim \sigma_{\text{hc}} \left( \sigma_{\text{sm}}^{-1} \|\mathbf{f}\|_{L'(\Omega, \mathbb{R}^d)} + \sigma_{\text{de}}^{|r-2|(\tilde{r}-1)} \left( \sigma_{\text{sm}}^{-1} \|\mathbf{f}\|_{L'(\Omega, \mathbb{R}^d)} \right)^{\frac{\tilde{r}-1}{r+1-\tilde{r}}} \right).$$

## Theorem. (Error estimate)

Let  $(\mathbf{u}, p) \in \mathbf{U} \times P$  and  $(\underline{\mathbf{u}}_h, p_h) \in \underline{\mathbf{U}}_{h,0}^k \times P_h^k$  solve the continuous and discrete weak formulations, respectively. Assume also

- ▶  $\mathbf{u} \in W^{k+2,r}(\mathcal{T}_h, \mathbb{R}^d)$ ,
- ▶  $\boldsymbol{\sigma}(\cdot, \nabla_s \mathbf{u}) \in W^{1,r'}(\Omega, \mathbb{R}_s^{d \times d}) \cap W^{(k+1)(\tilde{r}-1),r'}(\mathcal{T}_h, \mathbb{R}_s^{d \times d})$ ,
- ▶  $p \in W^{1,r'}(\Omega, \mathbb{R}) \cap W^{(k+1)(\tilde{r}-1),r'}(\mathcal{T}_h, \mathbb{R})$ .

Then,

$$\|\underline{\mathbf{u}}_h - \mathbf{I}_h^k \mathbf{u}\|_{\varepsilon,r,h} \lesssim C_1 h^{\frac{(k+1)(\tilde{r}-1)}{r+1-\tilde{r}}},$$

$$\|p_h - \pi_h^k p\|_{L^{r'}(\Omega, \mathbb{R})} \lesssim C_2 h^{\frac{(k+1)(\tilde{r}-1)^2}{r+1-\tilde{r}}},$$

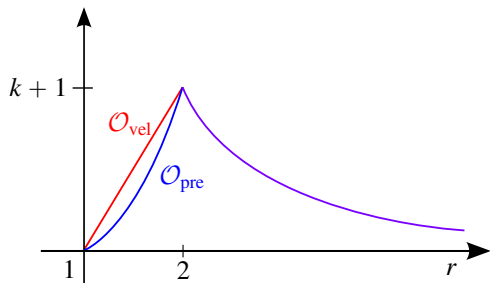
where  $C_1, C_2 \in [0, +\infty)$  depend only on  $\mathbf{u}, p, \mathbf{f}, \sigma_{hc}, \sigma_{sm}$ , and  $\sigma_{de}$ .

# Asymptotic convergence rates

Asymptotic convergence rates:

$$\mathcal{O}_{\text{vel}} := \frac{(k+1)(\tilde{r}-1)}{r+1-\tilde{r}} = \begin{cases} (k+1)(r-1) & \text{if } r < 2 \\ \frac{k+1}{r-1} & \text{if } r \geq 2 \end{cases},$$

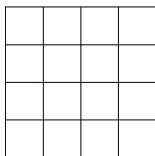
$$\mathcal{O}_{\text{pre}} := \frac{(k+1)(\tilde{r}-1)^2}{r+1-\tilde{r}} = \begin{cases} (k+1)(r-1)^2 & \text{if } r < 2 \\ \frac{k+1}{r-1} & \text{if } r \geq 2 \end{cases}.$$



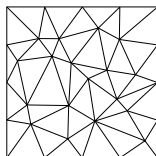
# Numerical results 2D

## SpaFEDTe library (2D and 3D)

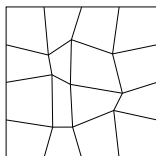
- ▶ We consider  $\Omega = (0, 1)^2$  and the following three mesh families.



Cartesian



distorted triangular



distorted Cartesian

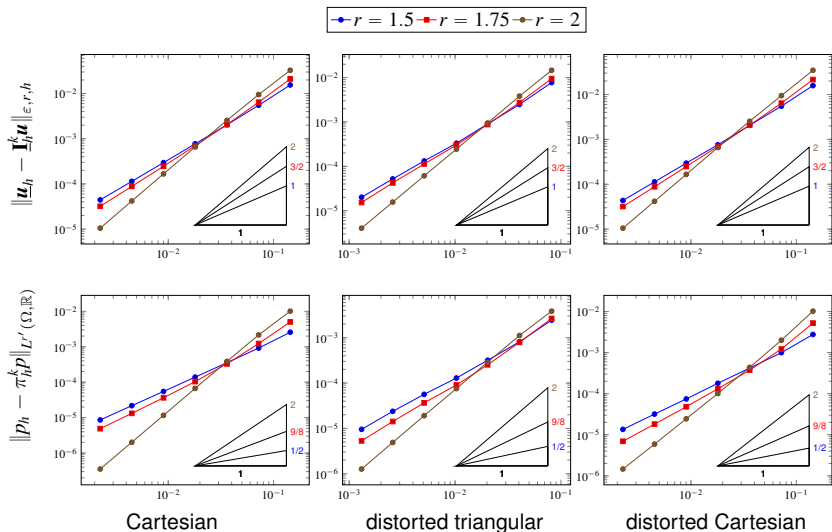
- ▶ For a well chosen  $f$ , the exact velocity  $\mathbf{u}$  and pressure  $p$  are given such that for all  $(x, y) \in \Omega$ ,

$$\mathbf{u}(x, y) = \left( \sin\left(\frac{\pi}{2}x\right) \cos\left(\frac{\pi}{2}y\right), -\cos\left(\frac{\pi}{2}x\right) \sin\left(\frac{\pi}{2}y\right) \right),$$

$$p(x, y) = \sin\left(\frac{\pi}{2}x\right) \sin\left(\frac{\pi}{2}y\right) - \frac{4}{\pi^2}.$$

# Numerical results 2D

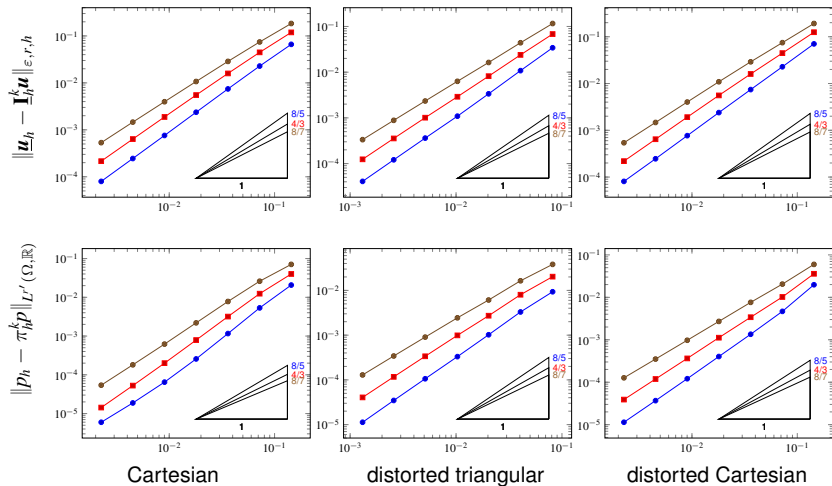
Results for  $k = 1$  and  $r \leq 2$ , so  $\mathcal{O}_{\text{vel}} = 2(r - 1)$  and  $\mathcal{O}_{\text{pre}} = 2(r - 1)^2$ .



# Numerical results 2D

Results for  $k = 1$  and  $r \geq 2$ , so  $\mathcal{O}_{\text{vel}} = \mathcal{O}_{\text{pre}} = \frac{2}{r-1}$ .

—•—  $r = 2.25$  —■—  $r = 2.5$  —●—  $r = 2.75$



# Next steps

## Next steps:

- ▶ In-depth analysis of the asymptotic convergence rates.
- ▶ Look for convergence by compactness.
- ▶ Extend the analysis to the Navier–Stokes equations.
- ▶ Moving to rheopecty and thixotropic fluids:  $r$  evolves over time.

# Thank you very much for your attention!

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