

IMPROVED ERROR ESTIMATES FOR HYBRID HIGH-ORDER DISCRETIZATIONS OF LERAY-LIONS PROBLEMS

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SERENA, INRIA – APRIL 1, 2021



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1. The Hybrid High-Order method
2. The Leray–Lions problem
3. Discretization with the HHO method
4. Numerical results

Hybrid High-Order (HHO)

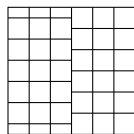
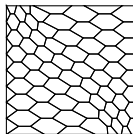
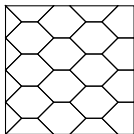
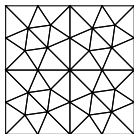
Hybrid: two kinds of unknowns located on the mesh and its skeleton.

High-Order: the unknowns live in broken polynomial spaces of degree $k \in \mathbb{N}$.

General features

This approach possesses several attractive features:

- ▶ Arbitrary **approximation order** ($k \geq 0$).
- ▶ Formulation valid for arbitrary **space dimension**.
- ▶ Seamless treatment of **nonconforming** mesh refinement.



- ▶ Moderate computational costs thanks to **static condensation**.
- ▶ **Inf-sup stable** discretizations.

Book.

Daniele A. Di Pietro and Jérôme Droniou.

The Hybrid High-Order method for polytopal meshes.

Design, Analysis, and Applications

Number 19 in Modeling, Simulation and Applications

Springer International Publishing, 2020

ISBN 978-3-030-37202-6 (Hardcover) 978-3-030-37203-3 (eBook)

DOI: [10.1007/978-3-030-37203-3](https://doi.org/10.1007/978-3-030-37203-3)

HAL preprint [hal-02151813](https://hal.archives-ouvertes.fr/hal-02151813)

Article.

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Improved error estimates for Hybrid High-Order discretizations of Leray–Lions problems

Calcolo, 2021 – Accepted for publication

arXiv preprint [2012.05122](https://arxiv.org/abs/2012.05122)

HAL preprint [hal-03049154](https://hal.archives-ouvertes.fr/hal-03049154)

Outline

1. The Hybrid High-Order method
2. The Leray–Lions problem
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The Leray–Lions problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}^*$, denote a bounded, connected polytopal open set with Lipschitz boundary $\partial\Omega$.

The Leray–Lions problem reads: Find $u : \Omega \rightarrow \mathbb{R}$ such that

$$\begin{aligned} -\nabla \cdot \sigma(\cdot, \nabla u) &= f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where,

- ▶ $u : \Omega \rightarrow \mathbb{R}$ is the **potential**,
- ▶ $f : \Omega \rightarrow \mathbb{R}$ is a volumetric **force** term,
- ▶ $\sigma : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is the **flux** function.

Assumptions on the flux function σ

Assumption. (Flux function)

There exists $p \in (1, 2]$ such that:

- ▶ $\sigma : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable.
- ▶ $\sigma(\cdot, \mathbf{0}) \in L^{p'}(\Omega)^d$ a.e. in Ω where $p' := \frac{p}{p-1}$.
- ▶ There is $\delta \in L^p(\Omega, [0, +\infty))$ and $\sigma_{\text{hc}}, \sigma_{\text{sm}} \in (0, +\infty)$ such that

$$|\sigma(\mathbf{x}, \boldsymbol{\tau}) - \sigma(\mathbf{x}, \boldsymbol{\eta})| \leq \sigma_{\text{hc}} (\delta(\mathbf{x})^p + |\boldsymbol{\tau}|^p + |\boldsymbol{\eta}|^p)^{\frac{p-2}{p}} |\boldsymbol{\tau} - \boldsymbol{\eta}|,$$

$$(\sigma(\mathbf{x}, \boldsymbol{\tau}) - \sigma(\mathbf{x}, \boldsymbol{\eta})) : (\boldsymbol{\tau} - \boldsymbol{\eta}) \geq \sigma_{\text{sm}} (\delta(\mathbf{x})^p + |\boldsymbol{\tau}|^p + |\boldsymbol{\eta}|^p)^{\frac{p-2}{p}} |\boldsymbol{\tau} - \boldsymbol{\eta}|^2,$$

for all $\boldsymbol{\tau}, \boldsymbol{\eta} \in \mathbb{R}^d$ and a.e. $\mathbf{x} \in \Omega$.

Examples of flux function

The following flux functions satisfy the previous assumption:

- ▶ The **p-Laplace** function s.t. for all $\tau \in \mathbb{R}^d$ and a.e. $x \in \Omega$,

$$\sigma(x, \tau) = |\tau|^{p-2} \tau.$$

- ▶ The **Carreau-Yasuda** function s.t. for all $\tau \in \mathbb{R}^d$ and a.e. $x \in \Omega$,

$$\sigma(x, \tau) = \mu(x) \left(\delta(x)^{a(x)} + |\tau|^{a(x)} \right)^{\frac{p-2}{a(x)}} \tau,$$

where $\mu : \Omega \rightarrow [\mu_-, \mu_+]$ and $a : \Omega \rightarrow [a_-, a_+]$ are measurable functions with $\mu_-, \mu_+, a_-, a_+ \in (0, +\infty)$.

Weak formulation

Assuming $f \in L^{p'}(\Omega)$, the **weak formulation** of the Leray–Lions problem reads: Find $u \in W_0^{1,p}(\Omega)$ such that

$$a(u, v) = \int_{\Omega} f v \quad \forall v \in W_0^{1,p}(\Omega),$$

where for all $v, w \in W_0^{1,p}(\Omega)$,

$$a(w, v) := \int_{\Omega} \sigma(\cdot, \nabla w) \cdot \nabla v.$$

Proposition. *(Well-posedness and a priori bound)*

There exists a **unique** solution $u \in W_0^{1,p}(\Omega)$ to the weak formulation. Additionally, the following a priori bound hold:

$$\|\nabla u\|_{L^p(\Omega)^d} \leq C \left(\sigma_{\text{sm}}, \|f\|_{L^{p'}(\Omega)}, \|\delta\|_{L^p(\mathcal{T}_h)}, \Omega, p \right).$$

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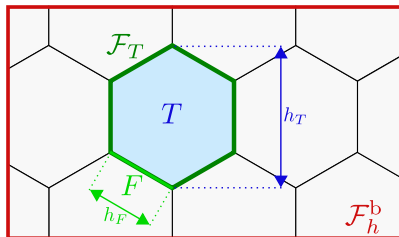
Mesh and notations

A polytopal mesh of Ω is a couple $\mathcal{M}_h := (\mathcal{T}_h, \mathcal{F}_h)$ defined such that

- ▶ \mathcal{T}_h is a finite collection of polytopal elements T with diameter h_T ,
- ▶ \mathcal{F}_h is a finite collection of hyperplanar faces F with diameter h_F ,
- ▶ $\bigcup_{T \in \mathcal{T}_h} \bar{T} = \bar{\Omega}$ and $\max_{T \in \mathcal{T}_h} h_T = h$,
- ▶ \mathcal{M}_h satisfies some geometrical requirements...

We also define the following subsets of \mathcal{F}_h :

- ▶ $\mathcal{F}_h^b := \{F \in \mathcal{F}_h : F \subset \partial\Omega\}$,
- ▶ $\mathcal{F}_T := \{F \in \mathcal{F}_h : F \subset \partial T\}$ for all $T \in \mathcal{T}_h$.



Discrete spaces and notations

Let $k \geq 0$ be the polynomial degree of the HHO method.

- ▶ For all $T \in \mathcal{T}_h$, we define the discrete **local** space:

$$\underline{U}_T^k := \mathbb{P}^k(T) \times \left(\prod_{F \in \mathcal{F}_T} \mathbb{P}^k(F) \right).$$

We use the discrete notation $\underline{v}_T := (v_T, (v_F)_{F \in \mathcal{F}_T}) \in \underline{U}_T^k$.

- ▶ We define the discrete **global** space:

$$\underline{U}_h^k := \prod_{T \in \mathcal{T}_h} \underline{U}_T^k.$$

We use the discrete notation $\underline{v}_h := (\underline{v}_T)_{T \in \mathcal{T}_h} \in \underline{U}_h^k$.

- ▶ We define the **interpolation** operator $\underline{I}_h^k : W^{1,1}(\Omega) \rightarrow \underline{U}_h^k$ s.t.

$$\underline{I}_h^k v := (\pi_T^k v|_T, (\pi_F^k v|_F)_{F \in \mathcal{F}_T})_{T \in \mathcal{T}_h}.$$

Spaces and norms of discrete unknowns

The **discrete space** containing potential unknowns is defined by

$$\underline{U}_{h,0}^k := \{ \underline{v}_h \in \underline{U}_h^k : \mathbf{v}_F = 0 \quad \forall F \in \mathcal{F}_h^b \}.$$

We endow $\underline{U}_{h,0}^k$ with the **norm** $\|\cdot\|_{1,p,h}$ defined s.t. for all $\underline{v}_h \in \underline{U}_{h,0}^k$,

$$\|\underline{v}_h\|_{1,p,h}^p := \sum_{T \in \mathcal{T}_h} \left(\|\nabla \mathbf{v}_T\|_{L^p(T)^d}^p + \sum_{F \in \mathcal{F}_T} h_F^{1-p} \|\mathbf{v}_F - \mathbf{v}_T\|_{L^p(F)}^p \right).$$

Discrete operators

For all $T \in \mathcal{T}_h$, we define:

- ▶ The **local gradient reconstruction** $\mathbf{G}_T^k : \underline{U}_T^k \longrightarrow \mathbb{P}^k(T)^d$ s.t. for all $\underline{v}_T \in \underline{U}_T^k$,

$$\int_T \mathbf{G}_T^k \underline{v}_T : \boldsymbol{\tau} = \int_T \nabla \mathbf{v}_T \cdot \boldsymbol{\tau} + \sum_{F \in \mathcal{F}_T} \int_F (\mathbf{v}_F - \mathbf{v}_T) \cdot (\boldsymbol{\tau} \mathbf{n}_{TF}) \quad \forall \boldsymbol{\tau} \in \mathbb{P}^k(T)^d.$$

- ▶ The **local potential reconstruction** $\mathbf{r}_T^{k+1} : \underline{U}_T^k \longrightarrow \mathbb{P}^{k+1}(T)$ s.t.

$$\int_T (\nabla \mathbf{r}_T^{k+1} \underline{v}_T - \mathbf{G}_T^k \underline{v}_T) \cdot \nabla w = 0 \quad \forall w \in \mathbb{P}^{k+1}(T),$$

$$\int_T \mathbf{r}_T^{k+1} \underline{v}_T = \int_T \mathbf{v}_T.$$

- ▶ The **boundary residual operator** $\Delta_{\partial T}^k : \underline{U}_T^k \rightarrow L^p(\partial T)$ s.t. for all $\underline{v}_T \in \underline{U}_T^k$,

$$(\Delta_{\partial T}^k \underline{v}_T)|_F := \frac{1}{h_T} [\pi_F^k(\mathbf{r}_T^{k+1} \underline{v}_T - \mathbf{v}_F) - \pi_T^k(\mathbf{r}_T^{k+1} \underline{v}_T - \mathbf{v}_T)] \quad \forall F \in \mathcal{F}_T.$$

Discrete weak formulation

The **discrete weak formulation** reads: Find $(\underline{u}_h, p_h) \in \underline{U}_{h,0}^k \times P_h^k$ s.t.

$$a_h(\underline{u}_h, \underline{v}_h) = \int_{\Omega} f v_h \quad \forall \underline{v}_h \in \underline{U}_{h,0}^k,$$

where, for all $\underline{w}_h, \underline{v}_h \in \underline{U}_h^k$,

$$a_h(\underline{w}_h, \underline{v}_h) := \sum_{T \in \mathcal{T}_h} \left(\int_T \boldsymbol{\sigma}(\cdot, \mathbf{G}_T^k \underline{w}_T) \cdot \mathbf{G}_T^k \underline{v}_T + h_T \int_{\partial T} \mathbf{S}_T(\cdot, \Delta_{\partial T}^k \underline{w}_T) \Delta_{\partial T}^k \underline{v}_T \right),$$

and where for all $T \in \mathcal{T}_h$, \mathbf{S}_T is a **local stabilization function** satisfying an assumption similar to that of $\boldsymbol{\sigma}$.

Assumptions on s_T

Assumption. (Stabilization functions)

There exists $\zeta \in L^p(\partial\mathcal{M}_h; [0, +\infty))$, where $\partial\mathcal{M}_h := \bigcup_{F \in \mathcal{F}_h} \bar{F}$, such that for all $T \in \mathcal{T}_h$:

- ▶ $S_T : \partial T \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable.
- ▶ $S_T(\mathbf{x}, 0) = 0$ for a.e. $\mathbf{x} \in \Omega$.
- ▶ For all $v, w \in \mathbb{R}$ and a.e. $\mathbf{x} \in \partial T$,

$$|S_T(\mathbf{x}, w) - S_T(\mathbf{x}, v)| \lesssim \sigma_{\text{hc}} (\zeta(\mathbf{x})^p + |w|^p + |v|^p)^{\frac{p-2}{p}} |w - v|,$$
$$(S_T(\mathbf{x}, w) - S_T(\mathbf{x}, v)) (w - v) \gtrsim \sigma_{\text{sm}} (\zeta(\mathbf{x})^p + |w|^p + |v|^p)^{\frac{p-2}{p}} |w - v|^2.$$

Stabilization functions that match the above assumption can be obtained setting for all $T \in \mathcal{T}_h$,

$$S_T(\mathbf{x}, w) = (\zeta(\mathbf{x})^p + |w|^p)^{\frac{p-2}{p}} w \quad \forall w \in \mathbb{R}, \quad \text{a.e. } \mathbf{x} \in \Omega.$$

Well-posedness and a priori bound

Lemma. *(Strong monotonicity of \mathbf{a}_h)*

For all $\underline{v}_h, \underline{w}_h \in \underline{U}_h^k$, setting $\underline{e}_h := \underline{v}_h - \underline{w}_h$,

$$\begin{aligned} \|\underline{e}_h\|_{1,p,h}^2 &\lesssim \sigma_{\text{sm}}^{-1} \left(\|\delta\|_{L^p(\Omega)}^p + \|\zeta\|_{L^p(\partial\mathcal{M}_h)}^p + \|\underline{v}_h\|_{1,p,h}^p + \|\underline{w}_h\|_{1,p,h}^p \right)^{\frac{2-p}{p}} \\ &\quad \times (\mathbf{a}_h(\underline{v}_h, \underline{e}_h) - \mathbf{a}_h(\underline{w}_h, \underline{e}_h)) \end{aligned}$$

Theorem. *(Well-posedness and a priori bound)*

There exists a **unique** solution $\underline{u}_h \in \underline{U}_{h,0}^k$ to the discrete weak formulation. Additionally, the following a priori bound hold:

$$\|\underline{u}_h\|_{1,p,h} \leq C \left(\sigma_{\text{sm}}, \|f\|_{L^{p'}(\Omega)}, \|\delta\|_{L^p(\mathcal{T}_h)}, \|\zeta\|_{L^p(\partial\mathcal{M}_h)}, \Omega, p \right).$$

Theorem. (Error estimate)

Let $u \in W_0^{1,p}(\Omega)$ and $\underline{u}_h \in \underline{U}_{h,0}^k$ solve the continuous and discrete weak formulations, respectively. Assume

- ▶ $u \in W^{k+2,p}(\mathcal{T}_h)$,
- ▶ $\sigma(\cdot, \nabla u) \in W^{1,p'}(\Omega)^d \cap W^{k+1,p'}(\mathcal{T}_h)^d$.

Then, setting $\underline{e}_h := \underline{u}_h - \underline{I}_h^k u$, we have

$$\|\underline{e}_h\|_{1,p,h} \leq h^{k+1} C_1 + \left[\sum_{T \in \mathcal{T}_h} \left[\min(\eta_T; 1)^{2-p} h_T^{(k+1)(p-1)} |u|_{W^{k+2,p}(T)}^{p-1} \right]^{p'} \right]^{\frac{1}{p'}} C_2,$$

where $C_1, C_2 \in [0, +\infty)$ depends only on u, f, σ , and where

$$\eta_T := h_T^{k+1} (|T|^{-\frac{1}{p}} |u|_{W^{k+2,p}(T)}) \mathfrak{D}_T^{-1}$$

$$\text{with } \mathfrak{D}_T := \min \left(\operatorname{ess\,inf}_T (\delta + |\nabla u|); \operatorname{ess\,inf}_{\partial T} \zeta \right).$$

Local convergence rates

For any $T \in \mathcal{T}_h$, we define the **local convergence rate** \mathcal{O}_T^k and determines it among 3 cases according to η_T :

- ▶ If $\eta_T \geq 1$ (**degenerate case**), then

$$\mathcal{O}_T^k = (k + 1)(p - 1).$$

- ▶ If $\eta_T \leq h_T^{k+1} (|T|^{-\frac{1}{p}} |u|_{W^{k+2,p}(T)})$ (**non-degenerate case**), then

$$\mathcal{O}_T^k = k + 1.$$

- ▶ Otherwise,

$$\mathcal{O}_T^k \in ((k + 1)(p - 1), k + 1).$$

The **non-degenerate case** is equivalent to $\mathfrak{D}_T \geq 1$. Moreover, we can choose $\zeta \geq 1$, so the non-degenerate case corresponds to

$$\operatorname{ess\,inf}_T (\delta + |\nabla u|) \geq 1.$$

Global convergence rates

The **global convergence rate** is naturally $\mathcal{O}_h^k := \min_{T \in \mathcal{T}_h} \mathcal{O}_T^k$. We can have an analogous disjunction of cases by defining $\eta_h = \max_{T \in \mathcal{T}_h} \eta_T$:

- ▶ If $\eta_h \geq 1$ (**degenerate case**), then

$$\mathcal{O}_h^k = (k + 1)(p - 1).$$

- ▶ If $\eta_h \leq h^{k+1} |u|_{W^{k+2,p}(\mathcal{T}_h)}$ (**non-degenerate case**), then

$$\mathcal{O}_h^k = k + 1.$$

- ▶ Otherwise,

$$\mathcal{O}_h^k \in ((k + 1)(p - 1), k + 1).$$

As at the local level, the **non-degenerate case** is equivalent to

$$\operatorname{ess\,inf}_{\Omega} (\delta + |\nabla u|) \geq 1.$$

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Numerical results

- ▶ We consider $\Omega = (0, 1)^2$ and a triangular mesh family.

- ▶ We choose the following Carreau-Yasuda flux function:

$$\sigma(\mathbf{x}, \boldsymbol{\tau}) = (\delta(\mathbf{x}) + |\boldsymbol{\tau}|)^{p-2} \boldsymbol{\tau} \quad \forall \boldsymbol{\tau} \in \mathbb{R}^2 \quad \text{a.e. } \mathbf{x} \in \Omega,$$

If $\delta = 0$ a.e. in Ω , then σ becomes the p -Laplace function.

- ▶ The force term f is given by u and σ .

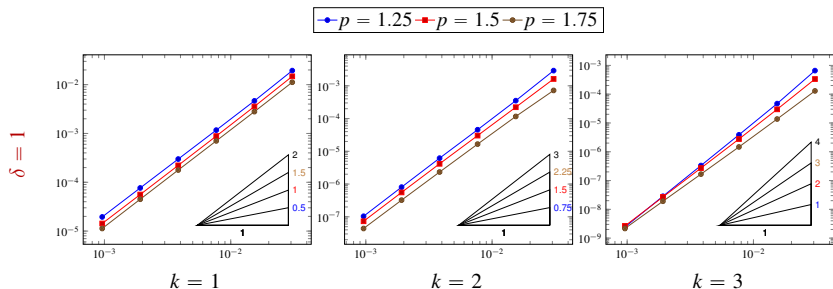
Non-degenerate flux

We consider $\delta \in \{1, 10^{-2}\}$ and define the potential such that

$$u(x, y) = \sin(\pi x) \sin(\pi y) \quad \forall (x, y) \in \Omega.$$

We have $\eta_h \simeq h^{k+1} (2^{\frac{k-1}{2}} \pi^k) \delta^{-1}$.

The convergence rate should switch from $(p-1)(k+1)$ to $(k+1)$ as h is small enough compared to δ .



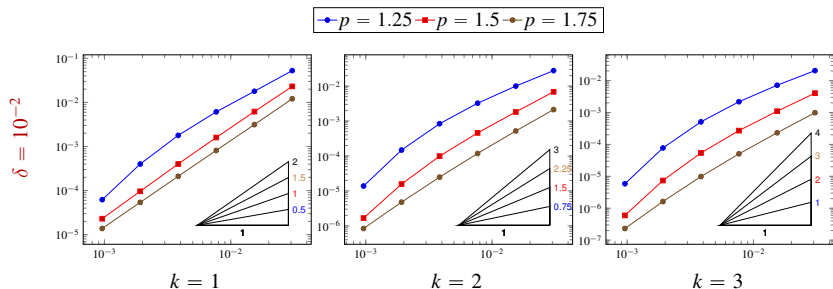
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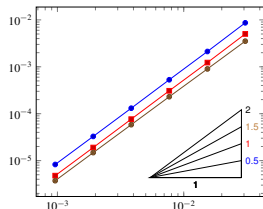
Non-degenerate potential

We set $\delta = 0$ and we define the potential such that

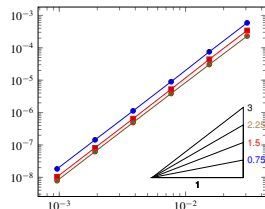
$$u(x, y) = \sin(\pi x) \sin(\pi y) + (\pi + 1)(x + y) \quad \forall (x, y) \in \Omega.$$

We have $\text{ess inf}_{\Omega} (\delta + |\nabla u|) = 1$, so we should observe a constant convergence rate of $(k + 1)$.

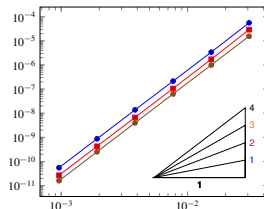
—•— $p = 1.25$ —•— $p = 1.5$ —•— $p = 1.75$



$k = 1$



$k = 2$



$k = 3$

Non-degenerate flux-potential couple

The potential is defined such that

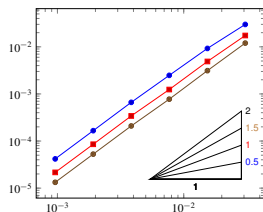
$$u(x, y) = \sin(\pi x) \sin(\pi y) \quad \forall (x, y) \in \Omega.$$

Let $(\mathbf{x}_i)_{1 \leq i \leq 5}$ be the points where ∇u vanishes, we set

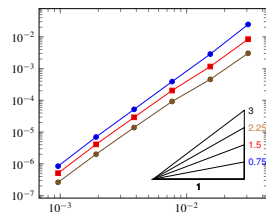
$$\delta(\mathbf{x}) = \sum_{i=1}^5 \mathbf{1}_{\{|\mathbf{x}-\mathbf{x}_i| < \frac{1}{3}\}} \exp\left(1 - \frac{1}{1 - 25|\mathbf{x} - \mathbf{x}_i|^2}\right) \quad \forall \mathbf{x} \in \Omega.$$

We infer $\text{ess inf}_{\Omega} (\delta + |\nabla u|) = 1$ which infer a $(k + 1)$ convergence rate.

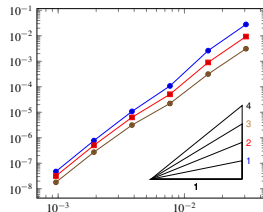
$$\text{---} \bullet \text{---} p = 1.25 \text{ ---} \blacksquare \text{---} p = 1.5 \text{ ---} \blacklozenge \text{---} p = 1.75$$



$k = 1$



$k = 2$



$k = 3$

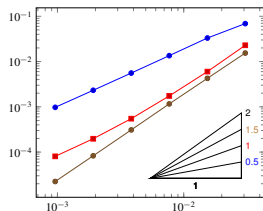
Degenerate problem

We set $\delta = 0$ and we choose the following potential:

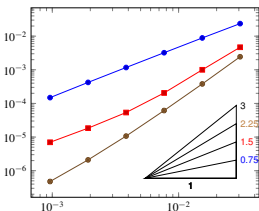
$$u(x, y) = \frac{1}{10} \exp \left(-10 \left(|x - 0.5|^{p + \frac{k+2}{4}} + |y - 0.5|^{p + \frac{k+2}{4}} \right) \right) \quad \forall (x, y) \in \Omega.$$

We have $\text{ess inf}_{\Omega} (\delta + |\nabla u|) = 0$, so $\eta_h = +\infty$ and the order of convergence expected is $(k+1)(p-1)$.

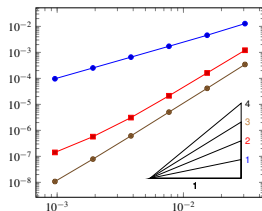
—•— $p = 1.25$ —■— $p = 1.5$ —●— $p = 1.75$



$k = 1$



$k = 2$



$k = 3$

Conclusion

Conclusion:

- ▶ We have presented and analysed a Hybrid High-Order scheme of arbitrary order k , for a non-linear model that generalises the p -Laplace equation with $p \in (1, 2]$.
- ▶ For a degenerate model we recover the known rates of convergence in $(k + 1)(p - 1)$, except when p is small where the convergence appears to be faster than expected.
- ▶ An optimal rate of $(k + 1)$, identical to the rate for **linear** models, is obtained when the model is not degenerate.
- ▶ These regimes are driven by a **dimensionless number**, and intermediate regimes are also identified.

Thank you very much for your attention!

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